OBJECTIVE
The members of a structure are subjected to internal forces like axial forces, shearing forces, bending and torsional moments while transferring the loads acting on it. Structural analysis deals with analysing these internal forces in the members of the structures. At the end of this course students will be conversant with classical method of analysis.

UNIT I DEFLECTION OF DETERMINATE STRUCTURES 9
Principles of virtual work for deflections – Deflections of pin-jointed plane frames and rigid plane frames – Willot diagram - Mohr’s correction

UNIT II MOVING LOADS AND INFLUENCE LINES (DETERMINATE & INDETERMINATE STRUCTURES) 9
Influence lines for reactions in statically determinate structures – influence lines for members forces in pin-jointed frames – Influence lines for shear force and bending moment in beam sections – Calculation of critical stress resultants due to concentrated and distributed moving loads.
Muller Breslau’s principle – Influence lines for continuous beams and single storey rigid frames – Indirect model analysis for influence lines of indeterminate structures – Beggs deformeter

UNIT III ARCHES 9
Arches as structural forms – Examples of arch structures – Types of arches – Analysis of three hinged, two hinged and fixed arches, parabolic and circular arches – Settlement and temperature effects.

UNIT IV SLOPE DEFLECTION METHOD 9
Continuous beams and rigid frames (with and without sway) – Symmetry and antisymmetry – Simplification for hinged end – Support displacements.

UNIT V MOMENT DISTRIBUTION METHOD 9
Distribution and carry over of moments – Stiffness and carry over factors – Analysis of continuous beams – Plane rigid frames with and without sway – Naylor’s simplification.

TUTORIAL 15 TOTAL : 60

TEXT BOOKS
3. Punmia B.C., Theory of Structures (SMTS ) Vol II laxmi Publishing Pvt ltd, New Delhi, 2004

REFERENCES
I UNIT – DEFLECTION OF DETERMINATE STRUCTURES

Theorem of minimum Potential Energy

Potential energy is the capacity to do work due to the position of body. A body of weight ‘W’ held at a height ‘h’ possess energy ‘Wh’. Theorem of minimum potential energy states that “Of all the displacements which satisfy the boundary conditions of a structural system, those corresponding to stable equilibrium configuration make the total potential energy a relative minimum”. This theorem can be used to determine the critical forces causing instability of the structure.

Law of Conservation of Energy

From physics this law is stated as “Energy is neither created nor destroyed”. For the purpose of structural analysis, the law can be stated as “If a structure and external loads acting on it are isolated, such that it neither receive nor give out energy, then the total energy of the system remain constant”. With reference to figure 2, internal energy is expressed as in equation (9). External work done \( W_e = -0.5 P \) dL. From law of conservation of energy \( U_i + W_e = 0 \). From this it is clear that internal energy is equal to external work done.

Principle of Virtual Work:

Virtual work is the imaginary work done by the true forces moving through imaginary displacements or vice versa. Real work is due to true forces moving through true displacements. According to principle of virtual work “The total virtual work done by a system of forces during a virtual displacement is zero”. Theorem of principle of virtual work can be stated as “If a body is in equilibrium under a Virtual force system and remains in equilibrium while it is subjected to a small deformation, the virtual work done by the external forces is equal to the virtual work done by the internal stresses due to these forces”. Use of this theorem for computation of displacement is explained by considering a simply supported beam AB, of span L, subjected to concentrated load P at C, as shown in Fig.6a. To compute deflection at D, a virtual load \( P' \) is applied at D after removing P at C. Work done is zero as the load is virtual. The load P is then applied at C, causing deflection \( \Delta_C \) at C and \( \Delta_D \) at D, as shown in Fig. 6b.

External work done \( W_e \) by virtual load \( P' \) is

\[
W_e = \frac{P \delta}{2}
\]

If the virtual load \( P' \) produces bending moment \( M' \), then the internal strain energy stored by \( M' \) acting on the real deformation \( d\theta \) in element \( dx \) over the beam equation (14)

\[
\int_0^L dU_i = \int_0^L \frac{M'd\theta}{2} \quad ; \quad U_i = \int_0^L \frac{M'M}{2EI} dx
\]
Where, \( M \) = bending moment due to real load \( P \). From principle of conservation of energy \( W_c = W_i \)

\[
\therefore \frac{P' \delta_D}{2} = \int_0^L \frac{M'M \, dx}{2 \, EI}
\]

If \( P' = 1 \) then

\[
\delta_D = \int_0^L \frac{M'M \, dx}{EI} \tag{16}
\]

Similarly for deflection in axial loaded trusses it can be shown that

\[
\delta = \sum_{0}^{n} \frac{P'P \, dx}{AE} \tag{17}
\]

Where,

- \( \delta \) = Deflection in the direction of unit load
- \( P' \) = Force in the \( i^{th} \) member of truss due to unit load
- \( P \) = Force in the \( i^{th} \) member of truss due to real external load
- \( n \) = Number of truss members
- \( L \) = length of \( i^{th} \) truss members.

Use of virtual load \( P' = 1 \) in virtual work theorem for computing displacement is called

**Unit Load Method**

**Castiglione’s Theorems:**
Castigliano published two theorems in 1879 to determine deflections in structures and redundant in statically indeterminate structures. These theorems are stated as:

**1st Theorem:** “If a linearly elastic structure is subjected to a set of loads, the partial derivatives of total strain energy with respect to the deflection at any point is equal to the load applied at that point”

\[
\frac{\partial U}{\partial \delta_j} = P_j \quad j = 1, 2, \ldots, N \quad (18)
\]

**2nd Theorem:** “If a linearly elastic structure is subjected to a set of loads, the partial derivatives of total strain energy with respect to a load applied at any point is equal to the deflection at that point”

\[
\frac{\partial U}{\partial P_j} = \delta_j \quad j = 1, 2, \ldots, N \quad (19)
\]

The first theorem is useful in determining the forces at certain chosen coordinates. The conditions of equilibrium of these chosen forces may then be used for the analysis of statically determinate or indeterminate structures. Second theorem is useful in computing the displacements in statically determinate or indeterminate structures.

**Betti’s Law:**

It states that if a structure is acted upon by two force systems I and II, in equilibrium separately, the external virtual work done by a system of forces II during the deformations caused by another system of forces I is equal to external work done by I system during the deformations caused by the II system

A body subjected to two system of forces is shown in Fig 7. \(W_{ij}\) represents work done by ith system of force on displacements caused by \(j^{th}\) system at the same point. Betti’s law can be expressed as \(W_{ij} = W_{ji}\), where \(W_{ji}\) represents the work done by \(j^{th}\) system on displacement caused by \(i^{th}\) system at the same point.
Trusses  Two Dimensional Structures

Three Dimensional Structures

Conditions of Equilibrium and Static Indeterminacy

A body is said to be under static equilibrium, when it continues to be under rest after application of loads. During motion, the equilibrium condition is called dynamic equilibrium. In two dimensional system, a body is in equilibrium when it satisfies following equation.

\[ \Sigma F_x = 0 ; \quad \Sigma F_y = 0 ; \quad \Sigma M_o = 0 \] --- 1.1

To use the equation 1.1, the force components along x and y axes are considered. In three dimensional system equilibrium equations of equilibrium are

\[ \Sigma F_x = 0 ; \quad \Sigma F_y = 0 ; \quad \Sigma F_z = 0 ; \]
\[ \Sigma M_x = 0 ; \quad \Sigma M_y = 0 ; \quad \Sigma M_z = 0 ; \] ---- 1.2

To use the equations of equilibrium (1.1 or 1.2), a free body diagram of the structure as a whole or of any part of the structure is drawn. Known forces and unknown reactions with assumed direction is shown on the sketch while drawing free body diagram. Unknown forces are computed using either equation 1.1 or 1.2.

Before analyzing a structure, the analyst must ascertain whether the reactions can be computed using equations of equilibrium alone. If all unknown reactions can be uniquely determined from the simultaneous solution of the equations of static equilibrium, the reactions of the structure are referred to as **statically determinate**. If they cannot be determined using equations of equilibrium alone then such structures are called **statically indeterminate structures**. If the number of unknown reactions are less than the number of equations of equilibrium then the structure is statically unstable.

The degree of indeterminacy is always defined as the difference between the number of unknown forces and the number of equilibrium equations available to solve for the unknowns. These extra forces are called redundants.
Indeterminacy with respect to external forces and reactions are called **externally indeterminate** and that with respect to internal forces are called **internally indeterminate**.

A general procedure for determining the degree of indeterminacy of two-dimensional structures are given below:

\[
\text{NUK} = \text{Number of unknown forces} \\
\text{NEQ} = \text{Number of equations available} \\
\text{IND} = \text{Degree of indeterminacy} \\
\text{IND} = \text{NUK} - \text{NEQ}
\]

**Indeterminacy of Planar Frames**

For entire structure to be in equilibrium, each member and each joint must be in equilibrium (Fig. 1.9)

\[
\text{NEQ} = 3\text{NM} + 3\text{NJ} \\
\text{NUK} = 6\text{NM} + \text{NR} \\
\text{IND} = \text{NUK} - \text{NEQ} = (6\text{NM} + \text{NR}) - (3\text{NM} + 3\text{NJ}) \\
\]  
\[
\text{IND} = 3\text{NM} + \text{NR} - 3\text{NJ} \quad \text{----- 1.3}
\]

**Three independent reaction components**

**Free body diagram of Members and Joints**

Degree of Indeterminacy is reduced due to introduction of internal hinge

\[
\text{NC} = \text{Number of additional conditions} \\
\text{NEQ} = 3\text{NM} + 3\text{NJ} + \text{NC} \\
\text{NUK} = 6\text{NM} + \text{NR} \\
\text{IND} = \text{NUK} - \text{NEQ} = 3\text{NM} + \text{NR} - 3\text{NJ} - \text{NC} \quad \text{-------- 1.3a}
\]

**Indeterminacy of Planar Trusses**

Members carry only axial forces

\[
\text{NEQ} = 2\text{NJ} \\
\text{NUK} = \text{NM} + \text{NR} \\
\text{IND} = \text{NUK} - \text{NEQ} \\
\text{IND} = \text{NM} + \text{NR} - 2\text{NJ} \quad \text{----- 1.4}
\]
**Indeterminacy of 3D FRAMES**
A member or a joint has to satisfy 6 equations of equilibrium

\[
\text{NEQ} = 6\text{NM} + 6\text{NJ} - \text{NC}
\]

\[
\text{NUK} = 12\text{NM} + \text{NR}
\]

\[
\text{IND} = \text{NUK} - \text{NEQ}
\]

\[
\text{IND} = 6\text{NM} + \text{NR} - 6\text{NJ} - \text{NC} \quad \cdots \quad 1.5
\]

**Indeterminacy of 3D Trusses**
A joint has to satisfy 3 equations of equilibrium

\[
\text{NEQ} = 3\text{NJ}
\]

\[
\text{NUK} = \text{NM} + \text{NR}
\]

\[
\text{IND} = \text{NUK} - \text{NEQ}
\]

\[
\text{IND} = \text{NM} + \text{NR} - 3\text{NJ} \quad \cdots \quad 1.6
\]

**Stable Structure:**
Another condition that leads to a singular set of equations arises when the body or structure is improperly restrained against motion. In some instances, there may be an adequate number of support constraints, but their arrangement may be such that they cannot resist motion due to applied load. Such situation leads to instability of structure. A structure may be considered as externally stable and internally stable.

**Externally Stable:**
- Supports prevents large displacements
- No. of reactions ≥ No. of equations

**Internally Stable:**
- Geometry of the structure does not change appreciably
  
  f) A 2D truss $\text{NM} \geq 2\text{Nj} - 3$ (NR ≥ 3)
  
  For a 3D truss $\text{NM} \geq 3\text{Nj} - 6$ (NR ≥ 3)

**Examples:**
Determine Degrees of Statical indeterminacy and classify the structures
NM=2; NJ=3; NR =4; NC=0
IND=3NM+NR-3NJ-NC
IND=3 x 2 + 4 - 3 x 3 -0 = 1
INDETERMINATE

NM=3; NJ=4; NR =5; NC=2
IND=3NM+NR-3NJ-NC
IND=3 x 3 + 5 - 3 x 4 -2 = 0
DETERMINATE

NM=3; NJ=4; NR =5; NC=2
IND=3NM+NR-3NJ-NC
IND=3 x 3 + 5 - 3 x 4 -2 = 0
DETERMINATE

NM=3; NJ=4; NR =3; NC=0
IND=3NM+NR-3NJ-NC
IND=3 x 3 + 3 - 3 x 4 -0 = 0
DETERMINATE

NM=1; NJ=2; NR =6; NC=2
IND=3NM+NR-3NJ-NC
IND=3 x 1 + 6 - 3 x 2 -2 = 1
INDETERMINATE

NM=1; NJ=2; NR =5; NC=1
IND=3NM+NR-3NJ-NC
IND=3 x 1 + 5 - 3 x 2 -1 = 1
INDETERMINATE

NM=1; NJ=2; NR =5; NC=1
IND=3NM+NR-3NJ-NC
IND=3 x 1 + 5 - 3 x 2 -1 = 1
INDETERMINATE
Each support has 6 reactions

NM=8; NJ=8; NR =24; NC=0
IND=6NM+NR-6NJ-NC
IND=6 x 8 + 24 – 6 x 8 -0 = 24
INDETERMINATE

Each support has 3 reactions

NM=18; NJ=15; NR =18; NC=0
IND=6NM+NR-6NJ-NC
IND=6 x 18 + 18 – 6 x 15 = 36
INDETERMINATE

Truss
NM=2; NJ=3; NR =4;
IND=NM+NR-2NJ
IND= 2 + 4 – 2 x 3 = 0
DETERMINATE

Truss
NM=14; NJ=9; NR =4;
IND=NM+NR-2NJ
IND= 14+ 4 – 2 x 9 = 0
Degree of freedom or Kinematic Indeterminacy

Members of structure deform due to external loads. The minimum number of parameters required to uniquely describe the deformed shape of structure is called “Degree of Freedom”. Displacements and rotations at various points in structure are the parameters considered in describing the deformed shape of a structure. In framed structure the deformation at joints is first computed and then shape of deformed structure. Deformation at intermediate points on the structure is expressed in terms of end deformations. At supports the deformations corresponding to a reaction is zero. For example hinged support of a two dimensional system permits only rotation and translation along x and y directions are zero. Degree of freedom of a structure is expressed as a number equal to number of free displacements at all joints. For a two dimensional structure each rigid joint has three displacements as shown in

In case of three dimensional structure each rigid joint has six displacement.

- Expression for degrees of freedom
  1. 2D Frames: NDOF = 3NJ – NR ≥3
  2. 3D Frames: NDOF = 6NJ – NR ≥6
  3. 2D Trusses: NDOF= 2NJ – NR ≥3
  4. 3D Trusses: NDOF = 3NJ – NR ≥6

Where, NDOF is the number of degrees of freedom

In 2D analysis of frames some times axial deformation is ignored. Then NAC>No. of axial condition is deducted from NDOF

Examples:

1.2 Determine Degrees of Kinematic Indeterminacy of the structures given below

a)
A Truss

Extensible
NJ=4; NR =5;
NDOF=3NJ-NR
NDOF=3 x 4 - 5 = 7
(θ₁, θ₂₁, θ₂₃, Δₙ₂, ε₁, ε₂)

Inextensible
NJ=4; NR =5; NAC=2
NDOF=3NJ-NR-NAC
NDOF=3 x 4 - 5-2= 5
(θ₁, θ₂₁, θ₂₃ , θ₃, Δ₂₃)

NJ=6; NR =3;
NDOF=2NJ-NR
NDOF=2 x 6 - 3 = 9
Virtual work is defined as the following line integral

\[ W = \int_C \mathbf{\bar{F}} \cdot \delta \mathbf{\bar{s}} \]

where

- \( C \) is the path or curve traversed by the object, keeping all constraints satisfied;
- \( \mathbf{\bar{F}} \) is the force vector;
- \( \delta \mathbf{\bar{s}} \) is the infinitesimal virtual displacement vector.

Virtual work is therefore a special case of mechanical work. For the work to be called virtual, the motion undergone by the system must be compatible with the system's constraints, hence the use of a virtual displacement.

One of the key ideas of Lagrangian mechanics is that the virtual work done by the constraint forces should be zero. This is a reasonable assumption, for otherwise a physical system might gain or lose energy simply by being constrained (imagine a bead on a stationary hoop moving faster and faster for no apparent reason)!
The idea of virtual work also plays a key role in interpreting D'Alembert's principle:

\[ \left( -\frac{\text{rdm}}{\text{inertial force}} \right) + df = 0 \]

Equilibrium of forces ("static" treatment)

\[ (\delta r)^T - (\text{rdm}) = \text{virtual work produced by inertia force} \]

\[ (\delta r)^T df = \text{virtual work produced by net applied force}. \]

Note: \( (\delta r)^T - \{\text{rdm} + df\} = 0 \)

Requirements on \( (\delta r) \):

- compatible with the kinematic constraints, but otherwise arbitrary
- instantaneous
- increasingly small

\[ \int (\delta r)^T (-\text{rdm} + df) = 0 \]

For a single body \( B_i \):

\[ \sum_{i=1}^{n} \int (\delta r)^T (-\text{rdm} + df) = 0 \]

For a system of \( n \) bodies \( B \):
Lagrange form of d’Alembert’s Principle

This formalism is convenient, as the constraint (non-working) loads disappear. (forces, torques) \( \rightarrow M\ddot{\mathbf{r}} + K\mathbf{r} = \mathbf{F}(t) \) where \( \mathbf{u} \) is the vector of independent degrees-of-freedom.

Example (i)

The motivation for introducing virtual work can be appreciated by the following simple example from statics of particles. Suppose a particle is in equilibrium under a set of forces \( F_{xi}, F_{yi}, F_{zi} \), \( i = 1, 2, ..., n \):

\[
\sum_{i=1}^{n} F_{xi} = 0 \\
\sum_{i=1}^{n} F_{yi} = 0 \\
\sum_{i=1}^{n} F_{zi} = 0 \quad (\varepsilon)
\]

Multiplying the three equations with the respective arbitrary constants \( \delta_x, \delta_y, \delta_z \):

\[
\delta_x \sum_{i=1}^{n} F_{xi} = 0 \\
\delta_y \sum_{i=1}^{n} F_{yi} = 0 \\
\delta_z \sum_{i=1}^{n} F_{zi} = 0 \quad (b)
\]

When the arbitrary constants \( \delta_x, \delta_y, \delta_z \) are thought of as virtual displacements of the particle, then the left-hand-sides of (b) represent the virtual work. The total virtual work is:

\[
\delta_x \sum_{i=1}^{n} F_{xi} + \delta_y \sum_{i=1}^{n} F_{yi} + \delta_z \sum_{i=1}^{n} F_{zi} = 0 \quad (c)
\]

Since the preceding equality is valid for arbitrary virtual displacements, it leads back to the equilibrium equations in (a). The equation (c) is called the principle of virtual work for a particle. Its use is equivalent to the use of many equilibrium equations.
Applying to a deformable body in equilibrium that undergoes compatible displacements and deformations, we can find the total virtual work by including both internal and external forces acting on the particles. If the material particles experience compatible displacements and deformations, the work done by internal stresses cancel out, and the net virtual work done reduces to the work done by the applied external forces. The total virtual work in the body may also be found by the volume integral of the product of stresses $\sigma$ and virtual strains $\delta \varepsilon$:

$$
\int_V \sigma \delta \varepsilon \, dV
$$

Thus, the principle of virtual work for a deformable body is:

$$
\text{External virtual work} = \int_V \sigma \delta \varepsilon \, dV
$$

This relation is equivalent to the set of equilibrium equations written for the particles in the deformable body. It is valid irrespective of material behaviour, and hence leads to powerful applications in structural analysis and finite element analysis.

Now consider a block on a surface

Applying formula (c) gives:

$$
\delta x (-m_1 \dot{x} + \dot{F}_x) + \delta y (-m_1 \dot{x} + \dot{F}_x + N - m_1 g) = 0
$$

leads to $-m_1 \dot{x} + \dot{F}_x = 0$

Observe virtual work formalism leads directly to Newton’s equation of motion in the kinematically allowable direction.

Example (ii)
Two bodies connected by a rotary joint.

Virtual work produced by these constraint loads:

\[ \delta w = (\delta r_{\text{joint}}) ^T F_{1,2} + (\delta r_{\text{joint}}) ^T F_{2,1} = (\delta r_{\text{joint}}) ^T \left( \frac{F_{1,2} + F_{2,1}}{0 \text{ (Newton)}} \right) \]

\( F_{1,2} \) and \( F_{2,1} \) drop out of the expression!

By assuming the contributions to virtual work produced by all forces in and an all system elements, the constraint loads disappear.

For multi-body system, the derivation of the equations of motion now becomes much more simple.

**Example 1:**
The particle moves from A to A'. The work done by the force \( \vec{F} \) is:

\[
dU = \vec{F} \cdot d\vec{r}
\]

Work is a scalar quantity & has units of ft-lb, in-lb, N-m, i.e., force times distance.

\[
dU = F \cdot dr \cdot \cos \alpha
\]

3 cases:

i) when \( F \) is in direction of \( dr \) (\( \alpha = 0^\circ \))

\[
dU = F \cdot dr
\]

ii) when \( F \) is opposite to \( dr \) (\( \alpha = 180^\circ \))

\[
dU = -F \cdot dr
\]

iii) when \( F \) is \( \perp \) to \( dr \) (\( \alpha = 90^\circ \))

\[
dU = 0
\]

Note: Internal forces do no work since these forces are always equal and opposite.

**Example 2**

**Example:** Force in a bar from a truss

Displacement occurs, \( A-A \) moves to \( A'-A' \)

\[
dU = -F \cdot x \cdot x = 0
\]

Work can also be done by moments

\[
dU = F \cdot dy \cdot \cos(\theta) = F \cdot dy
\]

For small angles \( dy = l \cdot d\theta \)

\[
dU = F \cdot l \cdot d\theta = M \cdot d\theta
\]

The physical quantity work is defined as the product of force times a conjugate displacement, i.e., a displacement in the same direction as the force we are considering. We are familiar with real work, i.e., the product of a real force and a real displacement, i.e., a force and a displacement that both actually occur. The situation is illustrated in Part 1 of the following figure:
We can extend the concept of real work to a definition of virtual work, which is the product of a real force and a conjugate displacement, either real or virtual. In Part 2 of the example shown above, we assume that the cantilever column loaded with force $P$ undergoes a virtual rotation of magnitude $\theta$ at its base. We compute the virtual work corresponding to this virtual displacement by summing the products of real forces times conjugate virtual displacements.

For this calculation, we must introduce unknown sectional forces at those locations where we have cut the structure to create the virtual displacement. In the example shown above, therefore, we have introduced bending moment at the base, $M_b$. For completeness, we would also have to introduce a shear force $V$ and an axial force $N$ at the base of the column, but, as we shall see, there is no component of virtual displacement conjugate to these forces. They have therefore not been shown in the example.

We calculate the virtual displacements of the structure corresponding to all known and unknown forces. For a rotation $\theta$ at the base, horizontal translation of the tip of the cantilever is $\theta \cdot L$. We then multiply force times displacement and sum these products to obtain the following expression for virtual work corresponding to the assumed virtual displacement:

$$ U = P \cdot L \cdot \theta - M_b \cdot \theta $$

We treat the virtual work done by force $M_b$ as negative since the direction of $M_b$ as drawn is opposite to the direction of the virtual rotation $\theta$.

The principle of virtual work states that a system of real forces is in equilibrium if and only if the virtual work performed by these forces is zero for all virtual displacements that are compatible with geometrical boundary conditions.
For the example given in the previous subsection, this implies that the virtual work of the simple cantilever, $U$, must be zero for the system to be in equilibrium:

$$U = P \cdot L \cdot \theta - M_b \cdot \theta = 0$$

Since $\theta$ is nonzero, it follows that $M_b = P \cdot L$, which is precisely the familiar expression for bending moment at the base of a cantilever loaded with force $P$ at its tip.

A more general mathematical statement of the principle of virtual work is as follows:

Let $Q_i$ be a set of real loads acting on a given structure
Let $R_i$ be the corresponding real support reactions
Let $M_i$, $V_i$, and $N_i$ be the sectional forces (bending moment, shear, and axial force) introduced at the locations where the structure has been cut to allow it to undergo a virtual displacement.

Let $\delta_{Q_i}$, $\delta_{R_i}$, $\delta_{M_i}$, $\delta_{V_i}$, and $\delta_{N_i}$ be virtual displacements compatible with the geometrical boundary conditions and conjugate to the forces defined previously.

Then the structure is in equilibrium if and only if:

$$\sum (Q_i \cdot \delta_{Q_i}) + \sum (R_i \cdot \delta_{R_i}) + \sum (M_i \cdot \delta_{M_i}) + \sum (V_i \cdot \delta_{V_i}) + \sum (N_i \cdot \delta_{N_i}) = 0$$

**Williot diagram**

The **Williot diagram** is a graphical method to obtain an approximate value for displacement of a structure which submitted to a certain load. The method consists of, from a graph representation of a structural system, representing the structure's fixed vertices as a single, fixed starting point and from there sequentially adding the neighbouring vertices' relative displacements due to strain.
Q1: Assume this is 30° even though not explicitly stated.

By inspection:

Vertical equilibrium:

\[ F_{BD} = 0 \]

Similarly, I think resolving perpendicular to the line AC:

\[ F_{EB} = 0 \]

At C:

\[ \sum F_y = 0 \quad F_{EC} \cos 60^\circ = 50 \text{ kN} \]

\[ \sum F_x = 0 \quad F_{BC} = F_{EC} \cos 30^\circ = 50 \sqrt{3} \text{ kN} \]

\[ F_{AB} = F_{BC} = 50 \sqrt{3} \text{ kN} \]

\[ F_{EO} = F_{OC} = 100 \text{ kN} \]

<table>
<thead>
<tr>
<th>Member</th>
<th>Force/KN</th>
<th>( S = \frac{F_L}{AE} ) /mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>50 \sqrt{3}</td>
<td>2.4</td>
</tr>
<tr>
<td>BC</td>
<td>50 \sqrt{3}</td>
<td>2.4</td>
</tr>
<tr>
<td>BD</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>CD</td>
<td>100</td>
<td>2.4</td>
</tr>
<tr>
<td>BE</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>DE</td>
<td>100</td>
<td>2.4</td>
</tr>
</tbody>
</table>

Steps:

1. Decide on scale
2. Draw O displacement point
II-UNIT-MOVING LOADS AND INFLUENCE LINES

In engineering, an influence line graphs the variation of a function (such as the shear felt in a structure member) at a specific point on a beam or truss caused by a unit load placed at any point along the structure. Some of the common functions studied with influence lines include reactions (the forces that the structure’s supports must apply in order for the structure to remain static), shear, moment, and deflection. Influence lines are important in the designing beams and trusses used in bridges, crane rails, conveyor belts, floor girders, and other structures where loads will move along their span. The influence lines show where a load will create the maximum effect for any of the functions studied.

Influence lines are both scalar and additive. This means that they can be used even when the load that will be applied is not a unit load or if there are multiple loads applied. To find the effect of any non-unit load on a structure, the ordinate results obtained by the influence line are multiplied by the magnitude of the actual load to be applied. The entire influence line can be scaled, or just the maximum and minimum effects experienced along the line. The scaled maximum and minimum are the critical magnitudes that must be designed for in the beam or truss.

In cases where multiple loads may be in effect, the influence lines for the individual loads may be added together in order to obtain the total effect felt by the structure at a given point. When adding the influence lines together, it is necessary to include the appropriate offsets due to the spacing of loads across the structure. For example, if it is known that load A will be three feet in front of load B, then the effect of A at x feet along the structure must be added to the effect of B at (x – 3) feet along the structure—not the effect of B at x feet along the structure. Many loads are distributed rather than concentrated. Influence lines can be used with either concentrated or distributed loadings. For a concentrated (or point) load, a unit point load is moved along the structure. For a distributed load of a given width, a unit-distributed load of the same width is moved along the structure, noting that as the load nears the ends and moves off the structure only part of the total load is carried by the structure. The effect of the distributed unit load can also be obtained by integrating the point load’s influence line over the corresponding length of the structures.
When designing a beam or truss, it is necessary to design for the scenarios causing the maximum expected reactions, shears, and moments within the structure members in order to ensure that no member will fail during the life of the structure. When dealing with dead loads (loads that never move, such as the weight of the structure itself), this is relatively easy because the loads are easy to predict and plan for. For live loads (any load that will be moved during the life of the structure, such as furniture and people), it becomes much harder to predict where the loads will be or how concentrated or distributed they will be throughout the life of the structure.

Influence lines graph the response of a beam or truss as a unit load travels across it. The influence line allows the designers to discover quickly where to place a live load in order to calculate the maximum resulting response for each of the following functions: reaction, shear, or moment. The designer can then scale the influence line by the greatest expected load to calculate the maximum response of each function for which the beam or truss must be designed. Influence lines can also be used to find the responses of other functions (such as deflection or axial force) to the applied unit load, but these uses of influence lines is less common.

**Influence Lines**

The major difference between shear and moment diagrams as compared to influence lines is that shear and bending moment diagrams show the variation of the shear and the moment over the entire structure for loads at a fixed position. An influence line for shear or moment shows the variation of the function at one section caused by a moving load.

Influence lines for functions of deterministic structures consists of a set of straight lines. The shape of influence lines for truss members are a bit more deceptive.

What we have looked at is quantitative influence lines. These have numerical values and can be computed. Qualitative influence lines are based on a principle by Heinrich Müller Breslau, which states:

"The deflected shape of a structure represents to some scale the influence line for a function such as reaction,"
shear or moment, if the function in question is allowed to act through a small distance."

In other words, is that the structure draws its own influence lines from the deflection curves. The shape of the influence lines can be created by deflecting the location in question by a moment, or shear or displacement to get idea of the behavior of the influence line. Realizing that the supports are zero values or poles.

Müller's principle for statically determinate structures is useful, but for indeterminated structures it is of great value. You can get an idea of the behavior of the shear and moment at a point in the beam.

Using influence lines to calculate values

From the previous examples of a twenty foot beam for the reactions, shear, and moment. We can use the values from the influence lines to calculate the shear and moment at a point.

\[ R_{Ay} = \sum (F_i) \times \text{Value of the influence line of } R_{Ay} \text{ @ location of the force} \]
\[ V_{11} = \sum (F_i) \times \text{Value of the influence line of } V_{11} \text{ @ location of the force} \]
\[ M_{11} = \sum (F_i) \times \text{Value of the influence line of } M_{11} \text{ @ location of the force} \]

If we are looking at the forces due to uniform loads over the beam at point. The shear or moment is equal to the area under the influence line times the distributed load.

\[ R_{Ay} = \sum (w_i) \times \text{Area of the influence line of } R_{Ay} \text{ over which } w \text{ covers} \]
\[ V_{11} = \sum (w_i) \times \text{Area of the influence line of } V_{11} \text{ over which } w \text{ covers} \]
\[ M_{11} = \sum (w_i) \times \text{Area of the influence line of } M_{11} \text{ over which } w \text{ covers} \]

For moving set of loads the influence lines can be used to calculate the maximum function. This can be done by moving the loads over the influence line find where they will generate the largest value for the particular point.

Panels or floating floor

The method can be extend to deal with floor joist and floating floors in which we deal with panels, which are simple beam elements acting on the floor joist. You will need to find the force as function of the intersection. You are going to find the moment and the shear as you move across the surface of the beam.

An example problem is used to show how this can be used to find the shear and moment at a point for a moving load. This technique is similar to that used in truss members.

Methods for constructing influence lines

There are three methods used for constructing the influence line. The first is to tabulate the influence values for multiple points along the structure, then use those points to create the influence line. The second is to determine the influence-line equations that apply to the structure, thereby solving for all points along the influence line in terms of \(x\), where \(x\) is the number of feet from the start of the structure to the point where the unit load is applied. The third method is called the Müller-Breslau principle. It creates a qualitative influence line. This influence line will still provide the designer with an accurate idea of where the unit load will produce the largest response of a function at the point being studied, but it cannot be used directly to calculate what the magnitude that response will be, whereas the influence lines produced by the first two methods can.

Influence-line equations

It is possible to create equations defining the influence line across the entire span of a structure. This is done by solving for the reaction, shear, or moment at the point A caused by a unit load placed at \(x\) feet along the structure instead of a
specific distance. This method is similar to the tabulated values method, but rather than obtaining a numeric solution, the outcome is an equation in terms of $x$.\(^5\)

It is important to understand where the slope of the influence line changes for this method because the influence-line equation will change for each linear section of the influence line. Therefore, the complete equation will be a **piecewise linear function** which has a separate influence-line equation for each linear section of the influence line.\(^5\)

**Müller-Breslau Principle**

The Müller-Breslau Principle can be utilized to draw **qualitative** influence lines, which are directly proportional to the actual influence line.\(^2\) Instead of moving a unit load along a beam, the Müller-Breslau Principle finds the deflected shape of the beam caused by first releasing the beam at the point being studied, and then applying the function (reaction, shear, or moment) being studied to that point. The principle states that the influence line of a function will have a scaled shape that is the same as the deflected shape of the beam when the beam is acted upon by the function. In order to understand how the beam will deflect under the function, it is necessary to remove the beam’s capacity to resist the function. Below are explanations of how to find the influence lines of a simply supported, rigid beam:

- When determining the reaction caused at a support, the support is replaced with a roller, which cannot resist a vertical reaction. Then an upward (positive) reaction is applied to the point where the support was. Since the support has been removed, the beam will rotate upwards, and since the beam is rigid, it will create a triangle with the point at the second support. If the beam extends beyond the second support as a cantilever, a similar triangle will be formed below the cantilevers position. This means that the reaction’s influence line will be a straight, sloping line with a value of zero at the location of the second support.

- When determining the shear caused at some point B along the beam, the beam must be cut and a roller-guide (which is able to resist moments but not shear) must be inserted at point B. Then, by applying a positive shear to that point, it can be seen that the left side will rotate down, but the right side will rotate up. This creates a discontinuous influence line which reaches zero at the supports and whose slope is equal on either side of the discontinuity. If point B is at a support, then the deflection between point B and any other supports will still create a triangle, but if the beam is cantilevered, then the entire cantilevered side will move up or down creating a rectangle.

- When determining the moment caused by at some point B along the beam, a hinge will be placed at point B, releasing it to moments but resisting shear. Then when a positive moment is placed at point B, both sides of the beam will rotate up. This will create a continuous influence line, but the slopes will be equal and opposite on either side of the hinge at point B. Since the beam is simply supported, its end supports (pins) cannot resist moment; therefore, it can be observed that the supports will never experience moments in a static situation regardless of where the load is placed.

The Müller-Breslau Principle can only produce qualitative influence lines. This means that engineers can use it to determine where to place a load to incur the maximum of a function, but the magnitude of that maximum cannot be calculated from the influence line. Instead, the engineer must use statics to solve for the functions value in that loading case.

For example, the influence line for the support reaction at A of the structure shown in Figure 1, is found by applying a unit load at several points (See Figure 2) on the structure and determining what the resulting reaction will be at A. This can be done by solving the support reaction $Y_A$ as a function of the position of a downward acting unit load. One such equation can be found by summing moments at Support B.

![Figure 1 - Beam structure for influence line example](image-url)
The graph of this equation is the influence line for the support reaction at A (See Figure 3). The graph illustrates that if the unit load was applied at A, the reaction at A would be equal to unity. Similarly, if the unit load was applied at B, the reaction at A would be equal to 0, and if the unit load was applied at C, the reaction at A would be equal to \(-e/L\).

Once an understanding is gained on how these equations and the influence lines they produce are developed, some general properties of influence lines for \textbf{statically determinate structures} can be stated.

1. **For a statically determinate structure** the influence line will consist of only straight line segments between critical ordinate values.
2. The influence line for a shear force at a given location will contain a translational discontinuity at this location. The summation of the positive and negative shear forces at this location is equal to unity.
3. Except at an internal hinge location, the slope to the shear force influence line will be the same on each side of the critical section since the bending moment is continuous at the critical section.
4. The influence line for a bending moment will contain a unit rotational discontinuity at the point where the bending moment is being evaluated.
5. To determine the location for positioning a single concentrated load to produce maximum magnitude for a particular function (reaction, shear, axial, or bending moment) place the load at the location of the maximum ordinate to the influence line. The value for the particular function will be equal to the magnitude of the concentrated load, multiplied by the ordinate value of the influence line at that point.
6. To determine the location for positioning a uniform load of constant intensity to produce the maximum magnitude for a particular function, place the load along those portions of the structure for which the ordinates to the influence line have the same algebraic sign. The value for the particular function will be equal to the magnitude of the uniform load, multiplied by the area under the influence diagram between the beginning and ending points of the uniform load.

There are two methods that can be used to plot an influence line for any function. In the first, the approach described above, is to write an equation for the function being determined, e.g., the equation for the shear, moment, or axial force induced at a point due to the application of a unit load at any other location on the structure. The second approach, which uses the \textbf{Müller Breslau Principle}, can be utilized to draw qualitative influence lines, which are directly proportional to the actual influence line.

The following examples demonstrate how to determine the influence lines for reactions, shear, and bending moments of beams and frames using both methods described above.

For example, the influence line for the support reaction at A of the structure shown in Figure 1, is found by applying a unit load at several points (See Figure 2) on the structure and determining what the resulting reaction will be at A. This
can be done by solving the support reaction $Y_A$ as a function of the position of a downward acting unit load. One such equation can be found by summing moments at Support B.

\[ \sum M_B = 0 \quad (\text{Assume counter-clockwise positive moment}) \]

\[ -Y_A(L) + 1(L-x) = 0 \]

\[ Y_A = \frac{(L-x)}{L} = 1 - \frac{(x}{L}) \]

The graph of this equation is the influence line for the support reaction at A (See Figure 3). The graph illustrates that if the unit load was applied at A, the reaction at A would be equal to unity. Similarly, if the unit load was applied at B, the reaction at A would be equal to 0, and if the unit load was applied at C, the reaction at A would be equal to $-e/L$.

### problem statement

Draw the influence lines for the reactions $Y_A$, $Y_C$, and the shear and bending moment at point B, of the simply supported beam shown by developing the equations for the respective influence lines.

- **Reaction $Y_A$**

The influence line for a reaction at a support is found by independently applying a unit load at several points on the structure and determining, through statics, what the resulting reaction at the support will be for each case. In this example, one such equation for the influence line of $Y_A$ can be found by summing moments around Support C.
\[
\sum M_C = 0 \hspace{1em} (\text{Assume counter-clockwise positive moment})
\]

- \(Y_A(25)+1(25-x) = 0\)
- \(Y_A = (25-x)/25 = 1 - (x/25)\)

The graph of this equation is the influence line for \(Y_A\) (See Figure 3). This figure illustrates that if the unit load is applied at A, the reaction at A will be equal to unity. Similarly, if the unit load is applied at B, the reaction at A will be equal to 1-(15/25)=0.4, and if the unit load is applied at C, the reaction at A will be equal to 0.

\[
Y_A = \frac{1}{25}x = 1 - \frac{x}{25}
\]

The fact that \(Y_A=1\) when the unit load is applied at A and zero when the unit load is applied at C can be used to quickly generate the influence line diagram. Plotting these two values at A and C, respectively, and connecting them with a straight line will yield the influence line for \(Y_A\). The structure is statically determinate, therefore, the resulting function is a straight line.

**Reaction at C**

The equation for the influence line of the support reaction at C is found by developing an equation that relates the reaction to the position of a downward acting unit load applied at all locations on the structure. This equation is found by summing the moments around support A.

\[
\sum M_A = 0 \hspace{1em} (\text{Assume counter-clockwise positive moment})
\]

- \(Y_C(25)-1(x) = 0\)
- \(Y_C = x/25\)

The graph of this equation is the influence line for \(Y_C\). This shows that if the unit load is applied at C, the reaction at C will be equal to unity. Similarly, if the unit load is applied at B, the reaction at C will be equal to 15/25=0.6. And, if the unit load is applied at A, the reaction at C will equal to 0.
The fact that $Y_C=1$ when the unit load is applied at $C$ and zero when the unit load is applied at $A$ can be used to quickly generate the influence line diagram. Plotting these two values at $A$ and $C$, respectively, and connecting them with a straight line will yield the influence line for $Y_C$. Notice, since the structure is statically determinate, the resulting function is a straight line.

### Shear at B

The influence line for the shear at point $B$ can be found by developing equations for the shear at the section using statics. This can be accomplished as follows:

a) if the load moves from $B$ to $C$, the shear diagram will be as shown in Fig. 6 below, this demonstrates that the shear at $B$ will equal $Y_A$ as long as the load is located to the right of $B$, i.e., $V_B = Y_A$. One can also calculate the shear at $B$ from the Free Body Diagram (FBD) shown in Fig. 7.

![Image](https://via.placeholder.com/150)

**Figure 6 - Shear diagram for load located between $B$ and $C$**

![Image](https://via.placeholder.com/150)

**Figure 7 - Free body diagram for section at $B$ with a load located between $B$ and $C$**

b) if the load moves from $A$ to $B$, the shear diagram will be as shown in Fig. 8, below, this demonstrates that the shear at $B$ will equal $-Y_C$ as long as the load is located to the left of $B$, i.e., $V_B = -Y_C$. One can also calculate the shear at $B$ from the FBD shown in Fig. 9.

![Image](https://via.placeholder.com/150)

**Figure 8 - Shear diagram for load located between $A$ and $B$**

![Image](https://via.placeholder.com/150)

**Figure 9 - Free body diagram for section at $B$ with a load located between $A$ and $B$**

The influence line for the shear at point $B$ is then constructed by drawing the influence line for $Y_A$ and negative $Y_C$. Then highlight the portion that represents the sides over which the load was moving. In this case, highlight the part from $B$ to $C$ on $Y_A$ and from $A$ to $B$ on $-Y_C$. Notice that at point $B$, the summation of the absolute values of the positive and negative shear is equal to 1.

![Image](https://via.placeholder.com/150)

**Figure 10 - Influence line for shear at point B**
• **Moment at B**

The influence line for the moment at point B can be found by using statics to develop equations for the moment at the point of interest, due to a unit load acting at any location on the structure. This can be accomplished as follows.

a) if the load is at a location between B and C, the moment at B can be calculated by using the FBD shown in Fig. 7 above, e.g., at B, $M_B = 15 Y_A$ - notice that this relation is valid if and only if the load is moving from B to C.

b) if the load is at a location between A and B, the moment at B can be calculated by using the FBD shown in Fig. 9 above, e.g., at B, $M_B = 10 Y_C$ - notice that this relation is valid if and only if the load is moving from A to B.

The influence line for the Moment at point B is then constructed by magnifying the influence lines for $Y_A$ and $Y_C$ by 15 and 10, respectively, as shown below. Having plotted the functions, $15 Y_A$ and $10 Y_C$, highlight the portion from B to C of the function $15 Y_A$ and from A to B on the function $10 Y_C$. These are the two portions what correspond to the correct moment relations as explained above. The two functions must intersect above point B. The value of the function at B then equals $(1 \times 10 \times 15)/25 = 6$. This represents the moment at B if the load was positioned at B.

![Figure 11 - Influence line for moment at point B](image)

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**Influence Lines**

**Qualitative Influence Lines using the Müller Breslau Principle**

• **Müller Breslau Principle**

The Müller Breslau Principle is another alternative available to qualitatively develop the influence lines for different functions. The Müller Breslau Principle states that the ordinate value of an influence line for any function on any structure is proportional to the ordinates of the deflected shape that is obtained by removing the restraint corresponding to the function from the structure and introducing a force that causes a unit displacement in the positive direction.

![Figure 1 - Beam structure to analyze](image)

For example, to obtain the influence line for the support reaction at A for the beam shown in Figure 1, above, remove the support corresponding to the reaction and apply a force in the positive direction that will cause a unit displacement in the direction of $Y_A$. The resulting deflected shape will be proportional to the true influence line for this reaction, i.e., for the support reaction at A. The deflected shape due to a unit displacement at A is shown below. Notice that the deflected shape is linear, i.e., the beam rotates as a rigid body without any curvature. This is true only for statically determinate systems.
Figure 2 - Support removed, unit load applied, and resulting influence line for support reaction at A

Similarly, to construct the influence line for the support reaction $Y_B$, remove the support at B and apply a vertical force that induces a unit displacement at B. The resulting deflected shape is the qualitative influence line for the support reaction $Y_B$.

Figure 3 - Support removed, unit load applied, and resulting influence line for support reaction at B

Once again, notice that the influence line is linear, since the structure is statically determinate. This principle will be now be extended to develop the influence lines for other functions.

- **Shear at s**

To determine the qualitative influence line for the shear at s, remove the shear resistance of the beam at this section by inserting a roller guide, i.e., a support that does not resist shear, but maintains axial force and bending moment resistance.

Figure 4 - Structure with shear capacity removed at s

Removing the shear resistance will then allow the ends on each side of the section to move perpendicular to the beam axis of the structure at this section. Next, apply a shear force, i.e., $V_{s-R}$ and $V_{s-L}$ that will result in the relative vertical displacement between the two ends to equal unity. The magnitude of these forces are proportional to the location of the section and the span of the beam. In this case,

$V_{s-L} = 1/16 \times 10 = 10/16 = 5/8$

$V_{s-R} = 1/16 \times 6 = 6/16 = 3/8$

The final influence line for $V_s$ is shown below.

Figure 5 - Influence line for shear at s
Shear just to the left side of B
The shear just to the left side of support B can be constructed using the ideas explained above. Simply imagine that section s in the previous example is moved just to the left of B. By doing this, the magnitude of the positive shear decreases until it reaches zero, while the negative shear increases to 1.

![Figure 6 - Influence line for shear just to the left of B](image)

Shear just to the right side of B
To plot the influence line for the shear just to the right side of support B, \( V_{b-R} \), release the shear just to the right of the support by introducing the type of roller shown in Fig. 7, below. The resulting deflected shape represents the influence line for \( V_{b-R} \). Notice that no deflection occurs between A and B, since neither of those supports were removed and hence the deflections at A and B must remain zero. The deflected shape between B and C is a straight line that represents the motion of a rigid body.

![Figure 7 - Structure with shear capacity removed at just to the right of B and the resulting influence line](image)

Moment at s
To obtain a qualitative influence line for the bending moment at a section, remove the moment restraint at the section, but maintain axial and shear force resistance. The moment resistance is eliminated by inserting a hinge in the structure at the section location. Apply equal and opposite moments respectively on the right and left sides of the hinge that will introduce a unit relative rotation between the two tangents of the deflected shape at the hinge. The corresponding elastic curve for the beam, under these conditions, is the influence line for the bending moment at the section. The resulting influence line is shown below.

![Figure 8 - Structure with moment capacity removed at s and the resulting influence line](image)

The values of the moments shown in Figure 8, above, are calculated as follows:

a. when the unit load is applied at s, the moment at s is \( Y_A \times 10 = \frac{3}{8} \times 10 = 3.75 \) (see the influence line for \( Y_A \), Figure 2, above, for the value of \( Y_A \) with a unit load applied at s)

b. when the unit load is applied at C, the moment at s is \( Y_A \times 10 = -\frac{3}{8} \times 10 = -3.75 \) (again, see the influence line for \( Y_A \) for the value of \( Y_A \) with a unit load applied at C)

Following the general properties of influence lines, given in the Introduction, these two values are plotted on the beam at the locations where the load is applied and the resulting influence line is constructed.
• Moment at B
The qualitative influence line for the bending moment at B is obtained by introducing a hinge at support B and applying a moment that introduces a unit relative rotation. Notice that no deflection occurs between supports A and B since neither of the supports were removed. Therefore, the only portion that will rotate is part BC as shown in Fig. 9, below.

Figure 9 - Structure with moment capacity removed at B and the resulting influence line

• Shear and moment envelopes due to uniform dead and live loads
The shear and moment envelopes are graphs which show the variation in the minimum and maximum values for the function along the structure due to the application of all possible loading conditions. The diagrams are obtained by superimposing the individual diagrams for the function based on each loading condition. The resulting diagram that shows the upper and lower bounds for the function along the structure due to the loading conditions is called the envelope.

The loading conditions, also referred to as load cases, are determined by examining the influence lines and interpreting where loads must be placed to result in the maximum values. To calculate the maximum positive and negative values of a function, the dead load must be applied over the entire beam, while the live load is placed over either the respective positive or negative portions of the influence line. The value for the function will be equal to the magnitude of the uniform load, multiplied by the area under the influence line diagram between the beginning and ending points of the uniform load.

For example, to develop the shear and moment envelopes for the beam shown in Figure 1, first sketch the influence lines for the shear and moment at various locations. The influence lines for $V_{a-R}$, $V_{b-L}$, $V_{b-R}$, $M_b$, $V_s$, and $M_s$ are shown in Fig. 10.

Figure 10 - Influence lines
These influence lines are used to determine where to place the uniform live load to yield the maximum positive and negative values for the different functions. For example:

![Diagram of a beam with supports and loads]

**Figure 11 -** Support removed, unit load applied, and resulting influence line for support reaction at A

- Maximum value for the positive reaction at A, assuming no partial loading, will occur when the uniform load is applied on the beam from A to B (load case 1)

**Figure 12 -** Load case 1

- The maximum negative value for the reaction at A will occur if a uniform load is placed on the beam from B to C (load case 2)

**Figure 13 -** Load case 2

- Load case 1 is also used for:
  - Maximum positive value of the shear at the right of support A
  - Maximum positive moment M_s

- Load case 2 is also used for:
  - Maximum positive value of the shear at the right of support B
  - Maximum negative moments at support B and M_s

- Load case 3 is required for:
  - Maximum positive reaction at B
  - Maximum negative shear on the left side of B

**Figure 14 -** Load case 3

- Load case 4 is required for the maximum positive shear force at section s
To develop the shear and moment envelopes, construct the shear and moment diagrams for each load case. The envelope is the area that is enclosed by superimposing all of these diagrams. The maximum positive and negative values can then be determined by looking at the maximum and minimum values of the envelope at each point.

Individual shear diagrams for each load case;

Figure 15 - Load case 4

Load case 5 is required for the maximum negative shear force at section s.

Figure 16 - Load case 5

To develop the shear and moment envelopes, construct the shear and moment diagrams for each load case. The envelope is the area that is enclosed by superimposing all of these diagrams. The maximum positive and negative values can then be determined by looking at the maximum and minimum values of the envelope at each point.

Individual shear diagrams for each load case;

Figure 17 - Individual shear diagrams

Superimpose all of these diagrams together to determine the final shear envelope.
Figure 18 - Resulting superimposed shear envelope
Individual moment diagrams for each load case;

Figure 19 - Individual moment diagrams
Superimpose all of these diagrams together to determine the final moment envelope.
Qualitative Influence Lines for a Statically Determinate Continuous Beam

problem statement

Draw the qualitative influence lines for the vertical reactions at the supports, the shear and moments at sections s1 and s2, and the shear at the left and right of support B of the continuous beam shown.

- Reactions at A, B, and C

Qualitative influence lines for the support reactions at A, B, and C are found by using the Müller Breslau Principle for reactions, i.e., apply a force which will introduce a unit displacement in the structure at each support. The resulting deflected shape will be proportional to the influence line for the support reactions.

The resulting influence lines for the support reactions at A, B, and C are shown in Figure 2, below.
Note: Beam BC does not experience internal forces or reactions when the load moves from A to h. In other words, influence lines for beam hC will be zero as long as the load is located between A and h. This can also be explained by the fact that portion hC of the beam is supported by beam ABh as shown in Figure 3, below.

Therefore, the force \( Y_h \) required to maintain equilibrium in portion hC when the load from h to C is provided by portion ABh. This force, \( Y_h \), is equal to zero when the load moves between A and h, and hence, no shear or moment will be induced in portion hC.

- Shear and moment at section S1 and S2

To determine the shear at \( s_1 \), remove the shear resistance of the beam at the section by inserting a support that does not resist shear, but maintains axial force and bending moment resistance (see the inserted support in Figure 4). Removing the shear resistance will allow the ends on each side of the section to move perpendicular to the beam axis of the structure at this section. Next, apply shear forces on each side of the section to induce a relative displacement between the two ends that will equal unity. Since the section is cut at the midspan, the magnitude of each force is equal to 1/2.

For the moment at \( s_1 \), remove the moment restraint at the section, but maintain axial and shear force resistance. The moment resistance is eliminated by inserting a hinge in the structure at the section location. Apply equal and opposite moments on the right and left sides of the hinge that will introduce a unit relative rotation between the two tangents of
the deflected shape at the hinge. The corresponding elastic curve for the beam, under these conditions, is the influence line for the bending moment at the section.

Figure 5 - Structure with moment capacity removed at s1 and resulting influence line

The value of the moment shown in Figure 5, above, is equal to the value of R_a when a unit load is applied at s_1, multiplied by the distance from A to s_1. M_{s1} = 1/2 \times 4 = 2.

The influence lines for the shear and moment at section s_2 can be constructed following a similar procedure. Notice that when the load is located between A and h, the magnitudes of the influence lines are zero for the shear and moment at s_1. The was explained previously in the discussion of the influence line for the support reaction at C (see Figures 2 and 3).

Figure 6 - Structure with shear capacity removed at s2 and resulting influence line

Figure 7 - Structure with moment capacity removed at s2 and resulting influence line

- Shear at the left and right of B

Since the shear at B occurs on both sides of a support, it is necessary to independently determine the shear for each side.

To plot the influence line for V_{b,L}, follow the instructions outlined above for plotting the influence line for the shear at s_1. To construct the shear just to the left of support B, imagine that the section s_1 has been moved to the left of B. In this case, the positive ordinates of the influence line between A and B will decrease to zero while the negative ordinates will increase to 1 (see Figure 8).
The influence line for the shear forces just to the right of support B, $V_{b-R}$, is represented by the resulting deflected shape of the beam induced by shear forces acting just to the right of support B. Notice that the portion of the beam from B to h moves as a rigid body (see explanation in the *Simple Beam with a Cantilever* example) while the influence line varies linearly from h to C. This is due to the fact that the deflection at C is zero and the assumption that the deflection of a statically determinate system is linear.

**Influence Lines**

**Calculation of Maximum and Minimum Shear Force and Moments on a Statically Determinate Continuous Beam**

**problem statement**

Determine the resulting forces for $R_A$, $R_B$, $R_C$, $M_{s1}$, $V_{s1}$, $M_{s2}$, $V_{s2}$, $V_{BL}$, and $V_{BR}$ under a uniform live load of 2 k/ft and a uniform dead load of 3 k/ft for the beam below.

*note: influence lines for this beam are developed in the *Statically Determinate Continuous Beam* example.*

- **Influence lines**

From the *Continuous Beam with a Hinge* example, the required influence lines for the structure are:
Calculate forces

In order to calculate the forces due to uniform dead and live loads on a structure, a relationship between the influence line and the uniform load is required. Referring to Figure 2, each segment \( dx \), of a uniform load \( w \), creates an equivalent concentrated load, \( dF = w \, dx \), acting a distance \( x \) from an origin.

From the general properties for influence lines, given in the introduction, it is known that the resulting value of the function for a force acting at a point is equivalent to the magnitude of the force, \( dF \), multiplied by the ordinate value, \( y \), of the influence line at the point of application.

In order to determine the effect of the uniform load, the effect of all series loads, \( dF \), must be determined for the beam. This is accomplished by integrating \( y \, dF \) over the length of the beam, i.e., \( \int w \, y \, dx = W \int y \, dx \). The integration of \( y \, dx \) equal to the area under the influence line. Thus, the value of the function caused by a uniform load is equal to the magnitude of the uniform load multiplied by the area under the influence line diagram.

In order to find the resulting minimum and maximum values for the reactions, shears, and moments required, create a table which contains the resulting positive and negative values for the areas enclosed by the influence lines for each function. The effect of the dead load is determined by multiplying the net area under the influence line by the dead load.
For the live load, multiply the respective positive and negative areas by the live load, yields to the positive and negative forces, respectively. The resulting maximum and minimum forces for dead load plus the effects of positive and negative live loads are then found by adding the respective values.

The resulting forces due to a uniformly distributed dead load = 3 k/ft and a live load = 2 k/ft applied to the beam above, are as follows:

<table>
<thead>
<tr>
<th>Force</th>
<th>Positive area under the influence line</th>
<th>Negative area under the influence line</th>
<th>Net area</th>
<th>Force due to DL</th>
<th>Positive force due to LL</th>
<th>Negative force due to LL</th>
<th>Maximum force (DL+LL)</th>
<th>Minimum force (DL-LL)</th>
</tr>
</thead>
<tbody>
<tr>
<td>R_A</td>
<td>4</td>
<td>-1</td>
<td>3</td>
<td>9</td>
<td>8</td>
<td>-2</td>
<td>17</td>
<td>7</td>
</tr>
<tr>
<td>R_B</td>
<td>10</td>
<td>-</td>
<td>10</td>
<td>30</td>
<td>20</td>
<td>-</td>
<td>50</td>
<td>30</td>
</tr>
<tr>
<td>R_C</td>
<td>3</td>
<td>-</td>
<td>3</td>
<td>9</td>
<td>6</td>
<td>-</td>
<td>15</td>
<td>9</td>
</tr>
<tr>
<td>M_{S1}</td>
<td>8</td>
<td>-4</td>
<td>4</td>
<td>12</td>
<td>16</td>
<td>8</td>
<td>28</td>
<td>4</td>
</tr>
<tr>
<td>V_{S1}</td>
<td>1</td>
<td>-2</td>
<td>-1</td>
<td>-3</td>
<td>2</td>
<td>4</td>
<td>-1</td>
<td>-7</td>
</tr>
<tr>
<td>M_{S2}</td>
<td>4.5</td>
<td>-</td>
<td>4.5</td>
<td>13.5</td>
<td>9</td>
<td>-</td>
<td>22.5</td>
<td>13.5</td>
</tr>
<tr>
<td>V_{S2}</td>
<td>0.75</td>
<td>-0.75</td>
<td>-</td>
<td>-</td>
<td>1.5</td>
<td>-1.5</td>
<td>1.5</td>
<td>-1.5</td>
</tr>
<tr>
<td>V_{B-R}</td>
<td>5</td>
<td>-</td>
<td>5</td>
<td>15</td>
<td>10</td>
<td>-</td>
<td>25</td>
<td>15</td>
</tr>
<tr>
<td>V_{B-L}</td>
<td>-</td>
<td>5</td>
<td>-5</td>
<td>-15</td>
<td>-</td>
<td>10</td>
<td>-15</td>
<td>-25</td>
</tr>
</tbody>
</table>

Column IV = Column II + Column III
Column V = Dead Load * Column IV
Column VI = Live Load * Column II
Column VII = Live Load * Column III
Column VIII = Column V + Column VI
Column IX = Column V + Column VII

---

**Influence Lines**

**Qualitative Influence Lines and Loading Patterns for an Multi-span Indeterminate Beam**

The Müller Breslau Principle, used previously to draw the influence lines for statically determinate structures, can also be extended to define the influence lines for indeterminate structures. This principle simply states that the influence line for a function is proportionally equivalent to the deflected shape of the structure when it undergoes a displacement as a result of the application of the function.

For indeterminate structures, an understanding of how complex structures deflect and react when acted upon by a force is required in order to draw accurate diagrams.
To determine the influence line for the support reaction at A, the Müller Breslau Principle requires the removal of the support restraint and the application of a positive unit deformation at this point that corresponds to the direction of the force. In this case, apply a unit vertical displacement in the direction of Y_A.

![Figure 1](image)

The resulting deflected shape, due to the application of the unit deformation, is then proportionally equivalent to the influence line for the support reaction at A. Notice that in statically indeterminate structures, the deflected shape is not a straight line, but rather a curve. The ordinates of the deflected shape decrease as the distance increases from the point of application of the unit deformation.

Similarly, for the other support reactions, remove the support restraint and apply a unit deformation in the direction of the removed restraint. For example, the influence line for the support reaction at C is obtained by removing the reaction at C and applying a unit displacement in the vertical direction at C. The resulting deflected shape is a qualitative representation of the influence line at R_C (see Figure 2).

![Figure 2](image)

Influence lines for the remaining support reactions are found in a similar manner.

### Influence lines for shears

For shear at a section, using the Müller Breslau Principle, the shear resistance at the point of interest is removed by introducing the type of support shown in Figure 3, below. Shear forces are applied on each side of the section in order to produce a relative displacement between the two sides which is equal to unity. The deflected shape of the beam under these conditions will qualitatively represent the influence line for the shear at the section. Notice that unlike the statically determinate structure, the magnitude of the shear force on the right and left can not easily be determined.

![Figure 3](image)

### Influence lines for moments

For the moment at a section, using the Müller Breslau Principle, the moment resistance at the point of interest is removed by introducing a hinge at the section as shown in Figure 4, below. Then a positive moment that introduces a relative unit rotation is applied at the section. The deflected shape of the beam under these conditions will qualitatively represent the influence line for the moment at the section.
For the moment at a support, the moment resistance is again removed by inserting a hinge at the support. This hinge only prevents the transfer of moments, so the vertical translation remains fixed due to the support. By applying negative moments that induces a relative rotation of unity at this section, a deflected shape is generated. Again, this deflected shape qualitatively represents the influence line for the moment at a support.

- **Loading cases for moment and shear envelopes**

Using the influence lines found above, illustrate the loading cases needed to calculate the maximum positive and negative $R_A$, $R_C$, $M_B$, $V_{S1}$, and $M_{S1}$.

The load cases are generated for the maximum positive and negative values by placing a distributed load on the spans where the algebraic signs of the influence line are the same. i.e., to get a maximum positive value for a function, place a distributed load where the influence line for the function is positive.
Qualitative Influence Lines and Loading Patterns for an Indeterminate Frame

**problem statement**

Using the Müller Breslau Principle, draw the influence lines for the moment and shear at the midspan of beam AB, and the moment at B in member BC. Draw the loading cases to give the maximum positive moment at the midspan of beam AB, the maximum and minimum shear at the midspan of beam AB, and the maximum negative moment at B in member BC in the indeterminate frame below.
- **Influence lines**

Influence line for moment at midspan of AB, and the loading case for maximum positive moment at this location.

The influence line for beam ABCD can be constructed by following the procedure outlined in the [Multi-span Indeterminate Beam example](#). To construct the rest of the influence line, make use of the fact that the angles between a column and a beam after deformation must be equal to that before deformation. In this example, these angles are 90°. Therefore, once the deflected shape of beam ABCD is determined, the deflected shape for the columns can be constructed by keeping the angles between the tangent of the deflect shape of the beam and the column equal to 90° (see Figure 2).

To get the maximum positive result for the moment, apply a distributed load at all locations where the value of the influence line is positive (see Figure 3).

Influence line for shear at the midspan of member AB, and the load case for maximum positive shear at this location.
Influence lines for shear at midspan of AB

Load case for maximum positive shear at midspan of AB

Load case for maximum negative shear at midspan of AB

Influence line for moment at B in member BC, and the load case for maximum negative moment at this location.

Influence lines for moment at B

Load case for maximum positive moment at B
III-UNIT ARCHES
THREE HINGED ARCHES

An arch is a curved beam in which horizontal movement at the support is wholly or partially prevented. Hence there will be horizontal thrust induced at the supports. The shape of an arch doesn’t change with loading and therefore some bending may occur.

Types of Arches
On the basis of material used arches may be classified into and steel arches, reinforced concrete arches, masonry arches etc.,

On the basis of structural behavior arches are classified as :

Three hinged arches:- Hinged at the supports and the crown.

Two hinged arches:- Hinged only at the support
A 3-hinged arch is a statically determinate structure. A 2-hinged arch is an indeterminate structure of degree of indeterminacy equal to 1. A fixed arch is a statically indeterminate structure. The degree of indeterminacy is 3.

Depending upon the type of space between the loaded area and the rib arches can be classified as open arch or closed arch (solid arch).

**Analysis of 3-hinged arches**

It is the process of determining external reactions at the support and internal quantities such as normal thrust, shear and bending moment at any section in the arch.

**Procedure to find reactions at the supports**

Step 1. Sketch the arch with the loads and reactions at the support.
Apply equilibrium conditions namely  \( \sum F_x = 0 \),  \( \sum F_y = 0 \) and  \( \sum M = 0 \)

Apply the condition that BM about the hinge at the crown is zero (Moment of all the forces either to the left or to the right of the crown).

Solve for unknown quantities.

A 3-hinged arch has a span of 30m and a rise of 10m. The arch carries UDL of 0.6 kN/m on the left half of the span. It also carries 2 concentrated loads of 1.6 kN and 1 kN at 5 m and 10 m from the ‘rt’ end. Determine the reactions at the support. (sketch not given).

\[
\sum F_x = 0 \\
H_A - H_B = 0 \\
H_A = H_B
\]

------ (1)

To find vertical reaction.

\[
\sum F_y = 0 \\
V_A + V_B = 0.6 \times 15 + 1 + 1.6
\]

\[= 11.6\]
\[ \sum M_A = 0 \]
\[- V_B \times 30 + 1.6 \times 25 + 1 \times 20 + (0.6 \times 15) = 7.5 \Rightarrow 0 \]

\[ V_B = 4.25 \text{kN} \]
\[ V_A = 4.25 = 11.6 \]
\[ A_A = 7.35 \text{kN} \]

To find horizontal reaction.

\[ M_C = 0 \]
\[- 1 \times 5 - 1.6 \times 10 + 4.25 \times 15 - H_B \times 10 = 0 \]

\[ H_B = 4.275 \text{kN} \]
\[ H_A = 4.275 \text{kN} \]

OR

\[ M_C = 0 \]
\[ 7.375 \times 15 - H_A \times 10 - (0.6 \times 15) \times 7.5 \]
\[ H_A = 4.275 \text{kN} \]
\[ H_B = 4.275 \text{kN} \]

To find total reaction

\[ V_A = 7.35 \text{kN} \]
\[ R_A \]
\[ V_B = 4.25 \text{kN} \]
\[ R_B \]
\[ H_A = 4.275 \text{kN} \]
\[ H_B = 4.275 \text{kN} \]

\[ R_A = \sqrt{H_A^2 + V_A^2} \]
\[ = \sqrt{4.275^2 + 7.35^2} \]
\[ = 8.5 \text{kN} \]

\[ \alpha_A = \tan^{-1} \left( \frac{V_A}{H_A} \right) = 59^\circ 82' \]

\[ R_B = \sqrt{H_B^2 + V_B^2} = 6.02 \text{kN} \]

\[ \alpha_B = \tan^{-1} \left( \frac{V_B}{H_B} \right) = 44.83 \]
A 3-hinged parabolic arch of span 50m and rise 15m carries a load of 10kN at quarter span as shown in figure. Calculate total reaction at the hinges.

\[ \sum F_x = 0 \]
\[ H_A = H_B \]

To find vertical reaction.
\[ \sum F_y = 0 \]
\[ V_A + V_B = 10 \]

\[ \sum M_A = 0 \]
\[ -V_B \times 50 + 10 \times 12.5 = 0 \]
\[ V_B = 2.5 \text{kN} \quad V_A = 7.5 \text{kN} \]

To find Horizontal reaction
\[ M_C = 0 \]
\[ V_B \times 25 - H_B \times 15 = 0 \]

To find total reaction.
\[ H_B = 4.17 \text{kN} = H_A \]
\[ R_A = \sqrt{4.17^2 + 7.5^2} \]
\[ R_A = 8.58 \text{ kN} \]
\[ \alpha_A = \tan^{-1}\left( \frac{V_A}{H_A} \right) = 60^\circ.92 \]
\[ R_B = \sqrt{H_A^2 + V_B^2} \]
\[ R_B = 4.86 \text{ kN} \]
\[ \alpha_B = \tan^{-1}\left( \frac{V_B}{H_B} \right) = 30^\circ.94 \]

Problem: Determine the reaction components at supports A and B for 3-hinged arch shown in fig.
To find Horizontal reaction
\[ \sum F_x = 0 \]
\[ H_A - H_B = 0 \]
\[ H_A = H_B \]  \hspace{1cm} (1)

To find vertical reaction.
\[ \sum F_y = 0 \]
\[ V_A + V_B = 180 + 10 \times 10 \]
\[ V_A + V_B = 280 \]  \hspace{1cm} (2)

\[ \sum M_A = 0 \]
\[ -V_B \times 24 + H_B \times 2.4 + 180 \times 18 + 10 \times 10 \times 5 = 0 \]
\[ 2.4H_B - 24V_B = -3740 \]  \hspace{1cm} (3)
\[ H_B - 10V_B = -155833 \]

\[ M_C = 0 \]
\[ -180 \times 8 - V_B \times 14 - H_B \times 4.9 = 0 \]
\[ H_B \times 4.9 - V_B \times 14 = -1440 \]  \hspace{1cm} (4)
\[ -H_B + 2.857V_B = +293.87 \]

Adding 2 and 3
Problem: A symmetrical 3-hinged parabolic arch has a span of 20m. It carries UDL of intensity 10 kNm over the entire span and 2 point loads of 40 kN each at 2m and 5m from left support. Compute the reactions. Also find BM, radial shear and normal thrust at a section 4m from left end take central rise as 4m.

\[-10V_B + 2.857V_B = -155833 + 293.87\]

\[V_B = 177kN\]

\[V_A = 103kN\]

\[H_B - 10 \times 177 = -155833\]

\[H_B = 211.67kN = H_A\]

\[\sum F_x = 0\]

\[H_A - H_B = 0\]

\[H_A = H_B\]  \hspace{1cm} \text{(1)}

\[\sum F_y = 0\]

\[V_A + V_B - 40 - 40 - 10 \times 20 = 0\]

\[V_A + V_B = 280\]  \hspace{1cm} \text{(2)}

\[\sum M_A = 0\]

\[+ 40 \times 2 + 40 \times 5 + (10 \times 20) - V_B \times 20 = 0\]

\[V_B = 114 \text{ kN}\]

\[V_A = 166 \text{ kN}\]

\[M_c = 0\]
\[-(10 \times 10)5 - H_B \times 4 + 114 \times 10 = 0\]

\[H_B = 160 \text{kN}\]

\[H_A = 160 \text{kN}\]

BM at M

\[= - 160 \times 2.56 + 166 \times 4 - 40 \times 2 - (10 \times 4)^2\]

\[= + 94.4 \text{kNm}\]

\[y = \frac{4hx}{L^2} (L - x)\]

\[= \frac{4 \times 4 \times 4}{20^2} (20 - 4)\]

\[y = 2.56 \text{m}\]

\[\tan \phi = \frac{4h}{L^2} (L - 2x)\]

\[= \frac{4 \times 4}{20^2} (20 - 2 \times 4)\]

\[\phi = 25^0.64\]

Normal thrust = \[N = + 160 \cos 25.64 + 86 \cos 64.36\]

\[= 181.46 \text{kN}\]

\[S = 160 \sin 25.64 - 86 \times \sin 64.36\]

\[S = -8.29 \text{kN}\]

IV UNIT  SLOPE DEFLECTION METHOD

The slope deflection method is a structural analysis method for beams and frames introduced in 1915 by George A. Maney. The slope deflection method was widely used for more than a decade until the moment distribution method was developed.

Introduction
By forming **slope deflection equations** and applying joint and shear equilibrium conditions, the rotation angles (or the slope angles) are calculated. Substituting them back into the slope deflection equations, member end moments are readily determined.

**Slope deflection equations**

The slope deflection equations express the member end moments in terms of rotations angles. The slope deflection equations of member $ab$ of flexural rigidity $EI_{ab}$ and length $L_{ab}$ are:

\[
M_{ab} = \frac{EI_{ab}}{L_{ab}} \left( 4\theta_a + 2\theta_b - 6\frac{\Delta}{L_{ab}} \right) \\
M_{ba} = \frac{EI_{ab}}{L_{ab}} \left( 2\theta_a + 4\theta_b - 6\frac{\Delta}{L_{ab}} \right)
\]

where $\theta_a$, $\theta_b$ are the slope angles of ends a and b respectively, $\Delta$ is the relative lateral displacement of ends a and b. The absence of cross-sectional area of the member in these equations implies that the slope deflection method neglects the effect of shear and axial deformations.

The slope deflection equations can also be written using the stiffness factor $K = \frac{I_{ab}}{L_{ab}}$ and the chord rotation $\psi$:

\[
\psi = \frac{\Delta}{L_{ab}}:
\]

\[
M_{ab} = 2EK \left( 2\theta_a + \theta_b - 3\psi \right) \\
M_{ba} = 2EK \left( \theta_a + 2\theta_b - 3\psi \right)
\]

**Derivation of slope deflection equations**

When a simple beam of length $L_{ab}$ and flexural rigidity $EI_{ab}$ is loaded at each end with clockwise moments $M_{ab}$ and $M_{ba}$, member end rotations occur in the same direction. These rotation angles can be calculated using the unit dummy force method or the moment-area theorem.

\[
\theta_a - \frac{\Delta}{L_{ab}} = \frac{L_{ab}}{3EI_{ab}} M_{ab} - \frac{L_{ab}}{6EI_{ab}} M_{ba} \\
\theta_b - \frac{\Delta}{L_{ab}} = - \frac{L_{ab}}{6EI_{ab}} M_{ab} + \frac{L_{ab}}{3EI_{ab}} M_{ba}
\]

Rearranging these equations, the slope deflection equations are derived.

**Equilibrium conditions**

**Joint equilibrium**

Joint equilibrium conditions imply that each joint with a degree of freedom should have no unbalanced moments i.e. be in equilibrium. Therefore,

\[
\Sigma \left( M^f + M_{\text{member}} \right) = \Sigma M_{\text{joint}}
\]
Here, $M_{\text{member}}$ are the member end moments, $M'$ are the fixed end moments, and $M_{\text{joint}}$ are the external moments directly applied at the joint.

Shear equilibrium

When there are chord rotations in a frame, additional equilibrium conditions, namely the shear equilibrium conditions need to be taken into account.

DISPLACEMENT METHOD OF ANALYSIS:

SLOPE DEFLECTION EQUATIONS

- General Case
- Stiffness Coefficients
- Stiffness Coefficients Derivation
- Fixed-End Moments
- Pin-Supported End Span
- Typical Problems
- Analysis of Beams
- Analysis of Frames: No Sidesway
- Analysis of Frames: Sidesway
Slope – Deflection Equations

settlement = \Delta_j
Degrees of Freedom

1 DOF: $\theta_A$

2 DOF: $\theta_A, \theta_B$
Stiffness

\[ k_{AA} = \frac{4EI}{L} \]

\[ k_{BA} = \frac{2EI}{L} \]
\[ k_{BB} = \frac{4EI}{L} \]

\[ k_{AB} = \frac{2EI}{L} \]
Fixed-End Forces

- Fixed-End Moments: Loads

\[
\frac{PL}{8}, \quad \frac{PL}{8}, \quad \frac{P}{2}, \quad \frac{P}{2}, \quad \frac{wL^2}{12}, \quad \frac{wL^2}{12}, \quad \frac{wL}{2}, \quad \frac{wL}{2}
\]
General Case

\[
P \quad w \quad \text{settlement} = \Delta_j
\]

\[
M_{ij} \quad M_{ji}
\]

\[
\theta_i \quad \theta_j
\]

\[
\psi
\]
\[ M_{ij} = \frac{4EI}{L} \theta_i + \frac{2EI}{L} \theta_j + (M^{Fij}_{ij})_\Delta + (M^{Fij}_{ij})_{\text{Load}} \]

\[ M_{ji} = \frac{2EI}{L} \theta_i + \frac{4EI}{L} \theta_j \]
Equilibrium Equations

\[ \sum M_j = 0: \quad -M_{ji} - M_{jk} + C_j = 0 \]
**Stiffness Coefficients**

\[ M_{ij} \]

\[ \theta_i \]

\[ \theta_j \]

\[ L \]

\[ k_{ii} = \frac{4EI}{L} \]

\[ k_{ji} = \frac{2EI}{L} \times \theta_i \]

\[ k_{ij} = \frac{2EI}{L} \]

\[ k_{jj} = \frac{4EI}{L} \times \theta_j \]
Matrix Formulation

\[ M_{ij} = \left(\frac{4EI}{L}\right)\theta_i + \left(\frac{2EI}{L}\right)\theta_j + (M^F_{ij}) \]

\[ M_{ji} = \left(\frac{2EI}{L}\right)\theta_i + \left(\frac{4EI}{L}\right)\theta_j + (M^F_{ji}) \]

\[
\begin{bmatrix}
M_{ij} \\
M_{ji}
\end{bmatrix} =
\begin{bmatrix}
(4EI/L) & (2EI/L) \\
(2EI/L) & (4EI/L)
\end{bmatrix}
\begin{bmatrix}
\theta_i \\
\theta_j
\end{bmatrix}
+ \begin{bmatrix}
M^F_{ij} \\
M^F_{ji}
\end{bmatrix}
\]

\[
[k] =
\begin{bmatrix}
k_{ii} & k_{ij} \\
\kappa_j & \kappa_{jj}
\end{bmatrix}
\]

Stiffness Matrix
\[
[M] = [K][\theta] + [FEM]
\]
\[
([M] - [FEM]) = [K][\theta]
\]
\[
[\theta] = [K]^{-1}([M] - [FEM])
\]

Stiffness matrix

Fixed-end moment matrix

\[
[D] = [K]^{-1}([Q] - [FEM])
\]

Displacement matrix

Force matrix
Stiffness Coefficients Derivation: Fixed-End Support

\[ M_i \]
\[ \theta_i \]
\[ M_j \]
\[ \frac{M_i + M_j}{L} \]
\[ \frac{M_i + M_j}{L} \]

\[ \frac{M_i}{EI} \]
\[ M_i = \frac{2M_j}{L} \quad \text{--- (1)} \]

\[ \sum M' = 0 : \quad -\left( \frac{M_i L}{2EI} \right) \left( \frac{L}{3} \right) + \left( \frac{M_j L}{2EI} \right) \left( \frac{2L}{3} \right) = 0 \]

\[ \sum F_y = 0 : \quad \theta_i - \left( \frac{M_i L}{2EI} \right) + \left( \frac{M_j L}{2EI} \right) = 0 \quad \text{--- (2)} \]

From (1) and (2):

\[ M_i = \frac{4EI}{L} \theta_i \]
\[ M_j = \frac{2EI}{L} \theta_i \]
Stiffness Coefficients Derivation: Pinned-End Support

Stiffness of Beam

\[ \frac{M_i}{L} \]

\[ \frac{M_i}{L} \]

\[ \frac{2L}{3} \]

\[ \frac{M_i}{L} \]

\[ \frac{M_i}{2EI} \]

\[ \frac{M_iL}{2EI} \]

\[ \frac{M_iL}{3EI} \]

\[ \theta_i = \left( \frac{M_iL}{3EI} \right) \]

\[ \theta_j = \left( \frac{-M_iL}{6EI} \right) \]

\[ \sum M'_j = 0: \quad \left( \frac{M_iL}{2EI} \right) \left( \frac{2L}{3} \right) - \theta_iL = 0 \]

\[ \sum F_y = 0: \quad \left( \frac{M_iL}{3EI} \right) - \left( \frac{M_iL}{2EI} \right) + \theta_j = 0 \]

\[ \theta_i = 1 = \left( \frac{M_iL}{3EI} \right) \quad \Rightarrow \quad M_i = \frac{3EI}{L} \]
Fixed end moment: Point Load

Real beam

Conjugate beam

\[ \sum F_y = 0: \quad -\frac{ML}{2EI} - \frac{ML}{2EI} + \frac{2PL^2}{16EI} = 0, \quad M = \frac{PL}{8} \]
Uniform load

Real beam

\[ w \]

Conjugate beam

\[ \sum F_y = 0: \quad -\frac{ML}{2EI} - \frac{ML}{2EI} + \frac{2wL^3}{24EI} = 0, \quad M = \frac{wL^2}{12} \]
\[ M_i = M_j \]

Real beam

Conjugate beam

\[ \sum M_B = 0: \quad -\Delta - \left( \frac{ML}{2EI} \right) \left( \frac{L}{3} \right) + \left( \frac{ML}{2EI} \right) \left( \frac{2L}{3} \right) = 0, \]

\[ M = \frac{6EI\Delta}{L^2} \]
Pin-Supported End Span: Simple Case

\[
\begin{align*}
\frac{4EI}{L} \theta_A + \frac{2EI}{L} \theta_B &= 0 = (4EI/L)\theta_A + (2EI/L)\theta_B + (FEM)_{AB} \quad \text{---- (1)} \\
\frac{2EI}{L} \theta_A + \frac{4EI}{L} \theta_B &= 0 = (2EI/L)\theta_A + (4EI/L)\theta_B + (FEM)_{BA} \quad \text{---- (2)} \\
2(2) - (1): \quad 2M_{BA} &= (6EI/L)\theta_B + 2(FEM)_{BA} - (FEM)_{BA} \\
M_{BA} &= (3EI/L)\theta_B + (FEM)_{BA} - \frac{(FEM)_{BA}}{2}
\end{align*}
\]
Pin-Supported End Span: With End Couple and Settlement

\[ M_{AB} = M_A = \frac{4EI}{L} \theta_A + \frac{2EI}{L} \theta_B + (M_{AB}^F)_{\text{load}} + (M_{AB}^F)_{\Delta} \quad \text{(1)} \]

\[ M_{BA} = \frac{2EI}{L} \theta_A + \frac{4EI}{L} \theta_B + (M_{BA}^F)_{\text{load}} + (M_{BA}^F)_{\Delta} \quad \text{(2)} \]

Eliminate \( \theta_A \) by \( \frac{2(2) - (1)}{2} \):

\[ M_{BA} = \frac{3EI}{L} \theta_B + \left[ (M_{BA}^F)_{\text{load}} - \frac{1}{2} (M_{AB}^F)_{\text{load}} \right] + \frac{1}{2} (M_{BA}^F)_{\Delta} + \frac{M_A}{2} \]
Fixed-End Moments

- Fixed-End Moments: Loads

\[ \frac{PL}{8} + \frac{1}{2} \left[ -\frac{PL}{8} \right] = \frac{3PL}{16} \]

\[ \frac{wL^2}{12} + \frac{1}{2} \left[ -\left( -\frac{wL^2}{12} \right) \right] = \frac{wL^2}{8} \]
Typical Problem

\[
M_{AB} = \frac{4EI}{L_1} \theta_A + \frac{2EI}{L_1} \theta_B + 0 + \frac{P_1L_1}{8}
\]

\[
M_{BA} = \frac{2EI}{L_1} \theta_A + \frac{4EI}{L_1} \theta_B + 0 - \frac{P_1L_1}{8}
\]

\[
M_{BC} = \frac{4EI}{L_2} \theta_B + \frac{2EI}{L_2} \theta_C + 0 + \frac{P_2L_2}{8} + \frac{wL_2^2}{12}
\]

\[
M_{CB} = \frac{2EI}{L_2} \theta_B + \frac{4EI}{L_2} \theta_C + 0 + \frac{-P_2L_2}{8} - \frac{wL_2^2}{12}
\]
\[ M_{BA} = \frac{2EI}{L_1} \theta_A + \frac{4EI}{L_1} \theta_B + 0 - \frac{P_1 L_1}{8} \]

\[ M_{BC} = \frac{4EI}{L_2} \theta_B + \frac{2EI}{L_2} \theta_C + 0 + \frac{P_2 L_2}{8} + \frac{wL_2^2}{12} \]

\[ \sum M_B = 0: \quad C_B - M_{BA} - M_{BC} = 0 \quad \rightarrow \quad \text{Solve for } \theta_B \]
Substitute $\theta_B$ in $M_{AB}$, $M_{BA}$, $M_{BC}$, $M_{CB}$

\[
M_{AB} = \frac{4EI}{L_1} \theta_A^0 + \frac{2EI}{L_1} \theta_B + 0 + \frac{P_1L_1}{8}
\]

\[
M_{BA} = \frac{2EI}{L_1} \theta_A^0 + \frac{4EI}{L_1} \theta_B + 0 - \frac{P_1L_1}{8}
\]

\[
M_{BC} = \frac{4EI}{L_2} \theta_B + \frac{2EI}{L_2} \theta_C + 0 + \frac{P_2L_2}{8} + \frac{wL_2^2}{12}
\]

\[
M_{CB} = \frac{2EI}{L_2} \theta_B + \frac{4EI}{L_2} \theta_C + 0 + \frac{-P_2L_2}{8} - \frac{wL_2^2}{12}
\]
The diagram illustrates a beam with forces and moments acting on it. The beam is divided into three sections: AB, BA, and BC.

- **Section AB**: The beam segment AB is acted upon by a moment $M_{AB}$ at point A and a force $P_1$ applied at point B. The beam is restrained by a supporting force $A_y$.

- **Section BA**: The segment BA is acted upon by a moment $M_{BA}$ at point B and a force $M_{BC}$ applied at point C. The beam is restrained by the force $B_{yL}$.

- **Section BC**: The segment BC is acted upon by a moment $M_{CB}$ at point C and a force $P_2$ applied at point B. The beam is restrained by the force $B_{yR}$.

The reactions at the ends of the beam are given by:

$$B_y = B_{yL} + B_{yR}$$

This equation represents the sum of the bending moments at the left and right ends of the beam. The forces and moments are illustrated in the diagram.
Example of Beams

Example 1

Draw the **quantitative shear**, **bending moment** diagrams and **qualitative deflected curve** for the beam shown. $EI$ is constant.
\[
[M] = [K][Q] + [\text{FEM}]
\]

\[
M_{AB} = \frac{4EI}{8} \theta_A + \frac{2EI}{8} \theta_B + \frac{(10)(8)}{8}
\]

\[
M_{BA} = \frac{2EI}{8} \theta_A + \frac{4EI}{8} \theta_B - \frac{(10)(8)}{8}
\]

\[
M_{BC} = \frac{4EI}{6} \theta_B + \frac{2EI}{6} \theta_c + \frac{(6)(6^2)}{30}
\]

\[
M_{CB} = \frac{2EI}{6} \theta_B + \frac{4EI}{6} \theta_c - \frac{(6)(6)^2}{20}
\]

\[\begin{align*}
\sum M_B &= 0: \quad -M_{BA} - M_{BC} = 0 \\
\left(\frac{4EI}{8} + \frac{4EI}{6}\right) \theta_B - 10 + \frac{(6)(6^2)}{30} &= 0 \\
\theta_B &= \frac{2.4}{EI}
\end{align*}\]

Substitute $\theta_B$ in the moment equations:

\[\begin{align*}
M_{AB} &= 10.6 \text{ kN} \cdot \text{m}, \quad M_{BC} = 8.8 \text{ kN} \cdot \text{m} \\
M_{BA} &= -8.8 \text{ kN} \cdot \text{m}, \quad M_{CB} = -10 \text{ kN} \cdot \text{m}
\end{align*}\]
$M_{AB} = 10.6 \text{kN}\cdot\text{m}, \quad M_{BC} = 8.8 \text{kN}\cdot\text{m}$

$M_{BA} = -8.8 \text{kN}\cdot\text{m}, \quad M_{CB} = -10 \text{kN}\cdot\text{m}$

$A_y = 5.23 \text{kN} \quad B_{yL} = 4.78 \text{kN} \quad B_{yR} = 5.8 \text{kN} \quad C_y = 12.2 \text{kN}$
**Example 2**

Draw the **quantitative shear**, **bending moment** diagrams and **qualitative deflected curve** for the beam shown. $EI$ is constant.
\[ [M] = [K][Q] + [\text{FEM}] \]

\[
M_{AB} = \frac{4EI}{8} \theta_A + \frac{2EI}{8} \theta_B + \frac{(10)(8)}{8} \quad (1)
\]

\[
M_{BA} = \frac{2EI}{8} \theta_A + \frac{4EI}{8} \theta_B - \frac{(10)(8)}{8} \quad (2)
\]

\[
M_{BC} = \frac{4EI}{6} \theta_B + \frac{2EI}{6} \theta_C + \frac{(6)(6^2)}{30} \quad (3)
\]

\[
M_{CB} = \frac{2EI}{6} \theta_B + \frac{4EI}{6} \theta_C - \frac{(6)(6)^2}{20} \quad (4)
\]

\[
2(2) - (1): \quad 2M_{BA} = \frac{6EI}{8} \theta_B - 30
\]

\[
M_{BA} = \frac{3EI}{8} \theta_B - 15 \quad (5)
\]
Substitute $\theta_A$ and $\theta_B$ in (5), (3) and (4):

$M_{BA} = -12.19 \text{ kN}\cdot\text{m}$

$M_{BC} = 12.19 \text{ kN}\cdot\text{m}$

$M_{CB} = -8.30 \text{ kN}\cdot\text{m}$
\[ M_{BA} = -12.19 \text{ kN\textbullet m}, \quad M_{BC} = 12.19 \text{ kN\textbullet m}, \quad M_{CB} = -8.30 \text{ kN\textbullet m} \]

\[ A_y = 3.48 \text{ kN} \quad B_{yL} = 6.52 \text{ kN} \quad B_{yR} = 6.65 \text{ kN} \quad C_y = 11.35 \text{ kN} \]
**Example 3**

Draw the **quantitative shear**, **bending moment** diagrams and **qualitative deflected curve** for the beam shown. \( EI \) is constant.
\[ M_{AB} = \frac{4(2EI)}{8} \theta_A + \frac{2(2EI)}{8} \theta_B + \frac{(10)(8)}{8} = \frac{10}{8} \quad \text{--- (1)} \]

\[ M_{BA} = \frac{2(2EI)}{8} \theta_A + \frac{4(2EI)}{8} \theta_B - \frac{(10)(8)}{8} = \frac{-10}{8} \quad \text{--- (2)} \]

\[ \frac{2(2) - (1)}{2} : \quad M_{BA} = \frac{3(2EI)}{8} \theta_B - \frac{(3/2)(10)(8)}{8} = \frac{15}{8} \quad \text{--- (2a)} \]

\[ M_{BC} = \frac{4(3EI)}{6} \theta_B + \frac{(4)(6^2)}{12} = \frac{12}{12} \quad \text{--- (3)} \]
\[ M_{BA} = \frac{3(2EI)}{8} \theta_B - \frac{(3/2)(10)(8)}{8} = -14.18 \text{ kN} \cdot \text{m} \]

\[ M_{BC} = \frac{4(3EI)}{6} \theta_B + \frac{(4)(6^2)}{12} = 14.18 \text{ kN} \cdot \text{m} \]

\[ -M_{BA} - M_{BC} = 0: \quad 2.75EI\theta_B = -12 + 15 = 3 \]

\[ \theta_B = 1.091 / EI \]

\[ M_{BA} = \frac{3(2EI)}{8} \left( \frac{1.091}{EI} \right) - 15 = -14.18 \text{ kN} \cdot \text{m} \]

\[ M_{BC} = \frac{4(3EI)}{6} \left( \frac{1.091}{EI} \right) - 12 = 14.18 \text{ kN} \cdot \text{m} \]

\[ M_{CB} = \frac{2(3EI)}{6} \theta_B - 12 = -10.91 \text{ kN} \cdot \text{m} \]
$M_{BA} = -14.18 \text{ kN} \cdot \text{m}$, $M_{BC} = 14.18 \text{ kN} \cdot \text{m}$, $M_{CB} = -10.91 \text{ kN} \cdot \text{m}$

$A_y = 3.23 \text{ kN}$

$B_{yL} = 6.73 \text{ kN}$

$B_{yR} = 12.55 \text{ kN}$

$C_y = 11.46 \text{ kN}$
10 kN

4 kN/m

A

3.23 kN

4 m

2EI

4 m

3EI

6 m

B

C

11.46 kN

10.91 kN•m

\[ V (kN) \]

3.23

-6.73

12.91

12.55

2.86

2EI

3EI

11.46 kN

6.77 + 12.55 = 19.32 kN

\[ M (kN\cdot m) \]

+12.91

+5.53

-14.18

-10.91

\[ \theta_B = \frac{1.091}{EI} \]

Deflected shape

\[ x (m) \]
Example 4

Draw the **quantitative shear**, **bending moment** diagrams and **qualitative deflected curve** for the beam shown. $EI$ is constant.
10 kN  12 kN•m  4 kN/m

A  2EI  3EI  C

1.5PL/8 = 15

wL²/12 = 12

4 m  4 m  6 m

\[ M_{BA} = \frac{3(2EI)}{8} \theta_B - 15 \quad \cdots (1) \]

\[ M_{BC} = \frac{4(3EI)}{6} \theta_B + 12 \quad \cdots (2) \]

\[ M_{CB} = \frac{2(3EI)}{6} \theta_B - 12 \quad \cdots (3) \]

Joint B: \[ -M_{BA} - M_{BC} - 12 = 0 \]

\[ -(0.75EI - 15) - (2EI \theta_B + 12) - 12 = 0 \]

\[ \theta_B = -\frac{3.273}{EI} \]

\[ M_{AB} = \frac{4(2EI)}{8} \theta_A + \frac{2(3EI)}{8} \theta_B + \frac{(10)(8)}{8} \]

\[ \theta_A = -\frac{7.21}{EI} \]

\[ M_{BA} = 0.75EI (\frac{3.273}{EI}) - 15 = -17.45 \text{ kN} \cdot \text{m} \]

\[ M_{BC} = 2EI (\frac{3.273}{EI}) + 12 = 5.45 \text{ kN} \cdot \text{m} \]

\[ M_{CB} = EI (\frac{3.273}{EI}) - 12 = -15.27 \text{ kN} \cdot \text{m} \]
\[ M_{BA} = 0.75EI\left(-\frac{3.273}{EI}\right) - 15 = -17.45 \text{ kN} \cdot \text{m} \]

\[ M_{BC} = 2EI\left(-\frac{3.273}{EI}\right) + 12 = 5.45 \text{ kN} \cdot \text{m} \]

\[ M_{CB} = EI\left(-\frac{3.273}{EI}\right) - 12 = -15.27 \text{ kN} \cdot \text{m} \]
Deflected shape

\[ \theta_A = -\frac{7.21}{EI} \]

\[ \theta_B = \frac{3.273}{EI} \]
Example 5

Draw the **quantitative shear**, **bending moment** diagrams, and **qualitative deflected curve** for the beam shown. Support B settles 10 mm, and $EI$ is constant. Take $E = 200 \, \text{GPa}$, $I = 200 \times 10^6 \, \text{mm}^4$. 

![Beam Diagram](image)
\[ M_{AB} = \frac{4(2EI)}{8} \theta_A + \frac{2(2EI)}{8} \theta_B + \frac{6(2EI)(0.01)}{8^2} + \frac{(10)(8)}{8} \quad (1) \]

\[ M_{BA} = \frac{2(2EI)}{8} \theta_A + \frac{4(2EI)}{8} \theta_B + \frac{6(2EI)(0.01)}{8^2} - \frac{(10)(8)}{8} \quad (2) \]

\[ M_{BC} = \frac{4(3EI)}{6} \theta_B + \frac{2(3EI)}{6} \theta_C - \frac{6(3EI)(0.01)}{6^2} + \frac{(6)(6^2)}{30} \quad (3) \]

\[ M_{CB} = \frac{2(3EI)}{6} \theta_B + \frac{4(3EI)}{6} \theta_C - \frac{6(3EI)(0.01)}{6^2} - \frac{(6)(6^2)}{30} \quad (4) \]
\[ M_{AB} = \frac{4(2EI)}{8} \theta_A + \frac{2(2EI)}{8} \theta_B + \frac{6(2EI)(0.01)}{8^2} + \frac{(10)(8)}{8} \quad \text{--- (1)} \]

\[ M_{BA} = \frac{2(2EI)}{8} \theta_A + \frac{4(2EI)}{8} \theta_B + \frac{6(2EI)(0.01)}{8^2} - \frac{(10)(8)}{8} \quad \text{--- (2)} \]

Substitute \( EI = (200 \times 10^6 \text{ kPa})(200 \times 10^{-6} \text{ m}^4) = 200 \times 200 \text{ kN} \cdot \text{ m}^2 \):

\[ M_{AB} = \frac{4(2EI)}{8} \theta_A + \frac{2(2EI)}{8} \theta_B + 75 + 10 \quad \text{--- (1)} \]

\[ M_{BA} = \frac{2(2EI)}{8} \theta_A + \frac{4(2EI)}{8} \theta_B + 75 - 10 \quad \text{--- (2)} \]

\[
\frac{2(2) - (1)}{2} : \quad M_{BA} = \frac{3(2EI)}{8} \theta_B + 75 - \frac{(75/2)}{2} - 10 - \frac{(10/2)}{2} - 12/2 = \quad \text{--- (2a)}
\]
\[ M_{BA} = (3/4)(2EI)\theta_B + 16.5 \]
\[ M_{BC} = (4/6)(3EI)\theta_B - 192.8 \]

\[ (3/4 + 2)EI\theta_B + 16.5 - 192.8 = 0 \]

\[ \theta_B = 64.109/ EI \]

Substitute \( \theta_B \) in (1):

\[ \theta_A = -129.06/ EI \]

Substitute \( \theta_A \) and \( \theta_B \) in (5), (3), (4):

\[ M_{BA} = 64.58 \text{ kN} \cdot \text{m}, \]
\[ M_{BC} = -64.58 \text{ kN} \cdot \text{m} \]
\[ M_{CB} = -146.69 \text{ kN} \cdot \text{m} \]
$M_{BA} = 64.58 \text{ kN}\cdot\text{m}$, 
$M_{BC} = -64.58 \text{ kN}\cdot\text{m}$

$M_{CB} = -146.69 \text{ kN}\cdot\text{m}$

$A_y = 11.57 \text{ kN}$ 
$B_{yL} = -1.57 \text{ kN}$

$B_{yR} = -29.21 \text{ kN}$

$C_y = 47.21 \text{ kN}$
Deflected shape

\[ \theta_A = \frac{-129.06}{EI} \]

\[ \theta_B = \frac{64.109}{EI} \]

Diagram showing:
- Load of 12 kN\(\cdot\)m at point A.
- Load of 10 kN at point B.
- Load of 6 kN/m applied at point C.
- Load of 146.69 kN\(\cdot\)m applied at point C.
- Vertical deflection at point B is 10 mm.

Shear force (V) diagram:
- 11.57 kN at 4 m.
- 1.57 kN at 4 m + 6 m.

Bending moment (M) diagram:
- 12 kN\(\cdot\)m at 4 m.
- 58.29 kN\(\cdot\)m at 4 m + 6 m.
- 64.58 kN\(\cdot\)m at 4 m + 6 m.

\[ 1.57 + 29.21 = 30.78 \text{ kN} \]

\[ \theta_A = \frac{-129.06}{EI} \]

\[ \theta_B = \frac{64.109}{EI} \]
Example 6

For the beam shown, support A settles 10 mm downward, use the slope-deflection method to
(a) Determine all the slopes at supports
(b) Determine all the reactions at supports
(c) Draw its quantitative shear, bending moment diagrams, and qualitative deflected shape. (3 points)
Take \( E = 200 \text{ GPa, } I = 50(10^6) \text{ mm}^4 \).
\[ M_{CB} = \frac{4(2EI)}{3} \theta_C \quad (1) \]

\[ M_{CA} = \frac{4(1.5EI)}{3} \theta_C + \frac{2(1.5EI)}{3} \theta_A + 4.5 + 100 \quad (2) \]

\[ M_{AC} = \frac{2(1.5EI)}{3} \theta_C + \frac{4(1.5EI)}{3} \theta_A - 4.5 + 100 \quad (3) \]

\[ \frac{2(2) - (2)}{2} : \quad M_{CA} = \frac{3(1.5EI)}{3} \theta_C + \frac{3(4.5)}{2} + \frac{100}{2} + \frac{12}{2} \quad (2a) \]
$M_{CB} = \frac{4(2EI)}{3} \theta_C \quad \cdots (1)$

$M_{CA} = \frac{3(1.5EI)}{3} \theta_C + \frac{3(4.5)}{2} + \frac{100}{2} + \frac{12}{2} \quad \cdots (2a)$

**Equilibrium equation:**

$M_{CB} + M_{CA} = 0$

$$\frac{(8+4.5)EI}{3} \theta_C + \frac{3(4.5)}{2} + \frac{100}{2} + \frac{12}{2} = 0$$

$$\theta_C = \frac{-15.06}{EI} = -0.0015 \, \text{rad}$$

Substitute $\theta_C$ in eq.(3)

$$12 = \frac{2(1.5EI)}{3} \left( \frac{-15.06}{EI} \right) + \frac{4(1.5EI)}{3} \theta_A - 4.5 + 100 \quad \cdots (3)$$

$$\theta_A = \frac{-34.22}{EI} = -0.0034 \, \text{rad}$$
\[
\theta_C = \frac{-15.06}{EI} = -0.0015 \text{ rad} \quad \theta_A = \frac{-34.22}{EI} = -0.0034 \text{ rad}
\]

\[
M_{BC} = \frac{2(2EI)}{3} \theta_C = \frac{2(2EI)}{3} \left( \frac{-15.06}{EI} \right) = -20.08 \text{ kN} \cdot \text{m}
\]

\[
M_{CB} = \frac{4(2EI)}{3} \theta_C = \frac{4(2EI)}{3} \left( \frac{-15.06}{EI} \right) = -40.16 \text{ kN} \cdot \text{m}
\]

\[
\frac{40.16 + 20.08}{3} = 20.08 \text{ kN}
\]
\( \theta_c = -0.0015 \text{ rad} \)

\( \theta_A = -0.0034 \text{ rad} \)

\( V \) (kN)

\( M \) (kN•m)

Deflected shape

-20.08
20.08
26.39
8.39

12

\( \theta_c = 0.0015 \text{ rad} \)

\( \theta_A = 0.0034 \text{ rad} \)
Example 7

For the beam shown, support A settles 10 mm downward, use the slope-deflection method to
(a) Determine all the slopes at supports
(b) Determine all the reactions at supports
(c) Draw its quantitative shear, bending moment diagrams, and qualitative deflected shape.
Take $E = 200$ GPa, $I = 50(10^6)$ mm$^4$. 

![Diagram of the beam with settlements and loads]
\[ 6(2EI) \Delta_C = \frac{4EI \Delta_C}{3} \]

\[ M_{BC} = \frac{2(2EI)}{3} \theta_C - \frac{4EI}{3} \Delta_C \quad \text{--- (1)} \]

\[ M_{CB} = \frac{4(2EI)}{3} \theta_C - \frac{4EI}{3} \Delta_C \quad \text{--- (2)} \]

\[ M_{CA} = \frac{4(1.5EI)}{3} \theta_C + \frac{2(1.5EI)}{3} \theta_A + EI \Delta_C + 4.5 + 100 \quad \text{--- (3)} \]

\[ M_{AC} = \frac{2(1.5EI)}{3} \theta_C + \frac{4(1.5EI)}{3} \theta_A + EI \Delta_C - 4.5 + 100 \quad \text{--- (4)} \]

\[ \frac{2(3)-(4)}{2} M_{CA} = \frac{3(1.5EI)}{3} \theta_c \]

\[ 12 \Delta_C = EI \Delta \]

\[ \frac{6(1.5) \Delta_C}{12} = EI \Delta \]

\[ 6(1.5 \times 200 \times 50)(0.01) \]

\[ \frac{3^2}{100 kN \cdot m} \]
\[ + \frac{EI}{2} \Delta_c + \frac{3(4.5)}{2} + \frac{12}{100} \frac{2}{(3a)} = 57 \]
• Equilibrium equation:

\[
(C_y)_{CB} = -\frac{(M_{BC} + M_{CB})}{3}
\]

\[
(C_y)_{CA} = \frac{M_{CA} + 12 + 18(1.5)}{3} = \frac{M_{CA} + 39}{3}
\]

\[
\sum M_C = 0: \quad M_{CB} + M_{CA} = 0 \quad --- (1*)
\]

\[
\sum C_y = 0: \quad (C_y)_{CB} + (C_y)_{CA} = 0 \quad --- (2*)
\]

Substitute in (1*) \[4.167 EI \theta_c - 0.8333 EI \Delta_c = -62.15 \quad --- (5)\]

Substitute in (2*) \[-2.5 EI \theta_c + 3.167 EI \Delta_c = -101.75 \quad --- (6)\]

From (5) and (6) \[\theta_c = -25.51 / EI = -0.00255 \text{ rad} \quad \Delta_c = -52.27 / EI = -5.227 \text{ mm}\]
\[ \theta_C = \frac{-25.51}{EI} = -0.00255 \text{ rad} \]

\[ \Delta_C = \frac{-52.27}{EI} = -5.227 \text{ mm} \]

Substitute \( \theta_C \) and \( \Delta_C \) in (4)

\[ \theta_A = \frac{-2.86}{EI} = -0.000286 \text{ rad} \]

Substitute \( \theta_C \) and \( \Delta_C \) in (1), (2) and (3a)

\[ M_{BC} = 35.68 \text{ kN} \cdot \text{m} \]

\[ M_{CB} = 1.67 \text{ kN} \cdot \text{m} \]

\[ M_{CA} = -1.67 \text{ kN} \cdot \text{m} \]

\[ B_y = 18 - 5.55 = 12.45 \text{ kN} \]

\[ A_y = \frac{18(4.5) - 12 - 35.68}{6} = 5.55 \text{ kN} \]
\[ \theta_c = -0.00255 \text{ rad} \]
\[ \Delta_c = -5.227 \text{ mm} \]
\[ \theta_A = -0.000286 \text{ rad} \]

\[ V (\text{kN}) = 12.45 \]
\[ M (\text{kN} \cdot \text{m}) = 1.67 \]
\[ -35.68 \]

\[ \Delta_c = 5.227 \text{ mm} \]
\[ \theta_c = 0.00255 \text{ rad} \]
\[ \theta_A = 0.000286 \text{ rad} \]
Example of Frame: No Sidesway
Example 6

For the frame shown, use the slope-deflection method to
(a) Determine the **end moments** of each member and **reactions** at supports
(b) Draw the **quantitative bending moment diagram**, and also draw the  
**qualitative deflected shape** of the entire frame.
• **Slope-Deflection Equations**

\[ M_{AB} = \frac{2(3EI)}{6} \theta_B + 30 \quad \ldots \quad (1) \]

\[ M_{BA} = \frac{4(3EI)}{6} \theta_B - 30 \quad \ldots \quad (2) \]

\[ M_{BC} = \frac{3(2EI)}{6} \theta_B + 36 + 18 \quad \ldots \quad (3) \]

• **Equilibrium equations**

\[ 10 - M_{BA} - M_{BC} = 0 \quad \ldots \quad (1^*) \]

Substitute (2) and (3) in (1*)

\[ 10 - 3EI\theta_B + 30 - 54 = 0 \]

\[ \theta_B = \frac{-14}{(3EI)} = -4.667 \frac{EI}{(3EI)} \]

*Substitute* \( \theta_B = -4.667 \frac{EI}{(3EI)} \) in (1) to (3)

\[ M_{AB} = 25.33 \text{ kN} \cdot \text{m} \]

\[ M_{BA} = -39.33 \text{ kN} \cdot \text{m} \]

\[ M_{BC} = 49.33 \text{ kN} \cdot \text{m} \]
\( M_{AB} = 25.33 \text{ kN}\cdot\text{m} \)

\( M_{BA} = -39.33 \text{ kN}\cdot\text{m} \)

\( M_{BC} = 49.33 \text{ kN}\cdot\text{m} \)

\( \theta_B = -4.667/EI \)

Bending moment diagram

Deflected curve
Example 7

Draw the **quantitative shear**, **bending moment** diagrams and **qualitative deflected curve** for the frame shown. $E = 200$ GPa.
\( \frac{PL}{8} = 18.75 \)

\[
\begin{align*}
M_{AB} &= \frac{4(2EI)}{5} \theta_A + \frac{2(2EI)}{5} \theta_B \\
M_{BA} &= \frac{2(2EI)}{5} \theta_A + \frac{4(2EI)}{5} \theta_B \\
M_{BC} &= \frac{4(4EI)}{6} \theta_B + \frac{2(4EI)}{6} \theta_C + 18.75 \\
M_{CB} &= \frac{2(4EI)}{6} \theta_B + \frac{4(4EI)}{6} \theta_C - 18.75 \\
M_{CD} &= \frac{3(EI)}{5} \theta_C \\
M_{CE} &= \frac{3(3EI)}{4} \theta_C + 10
\end{align*}
\]

\[
(\frac{8}{5} + \frac{16}{6})EI\theta_B + (\frac{8}{6})EI\theta_C = -18.75 \quad ---(1)
\]

\[
M_{BA} + M_{BC} = 0
\]

\[
(\frac{8}{6})EI\theta_B + (\frac{16}{6} + \frac{3}{5} + \frac{9}{4})EI\theta_C = 8.75 \quad ---(2)
\]

From (1) and (2): \( \theta_B = \frac{-5.29}{EI} \quad \theta_C = \frac{2.86}{EI} \)
Substitute \( \theta_B = -1.11/EI, \theta_c = -20.59/EI \) below

\[
M_{AB} = \frac{4(2EI)}{5} \theta_A + \frac{2(2EI)}{5} \theta_B \quad \rightarrow \quad M_{AB} = -4.23 \text{ kN} \cdot \text{m}
\]

\[
M_{BA} = \frac{2(2EI)}{5} \theta_A + \frac{4(2EI)}{5} \theta_B \quad \rightarrow \quad M_{BA} = -8.46 \text{ kN} \cdot \text{m}
\]

\[
M_{BC} = \frac{4(4EI)}{6} \theta_B + \frac{2(4EI)}{6} \theta_c + 18.75 \quad \rightarrow \quad M_{BC} = 8.46 \text{ kN} \cdot \text{m}
\]

\[
M_{CB} = \frac{2(4EI)}{6} \theta_B + \frac{4(4EI)}{6} \theta_c - 18.75 \quad \rightarrow \quad M_{CB} = -18.18 \text{ kN} \cdot \text{m}
\]

\[
M_{CD} = \frac{3(EI)}{5} \theta_c \quad \rightarrow \quad M_{CD} = 1.72 \text{ kN} \cdot \text{m}
\]

\[
M_{CE} = \frac{3(3EI)}{4} \theta_c + 10 \quad \rightarrow \quad M_{CE} = 16.44 \text{ kN} \cdot \text{m}
\]
\[ M_{AB} = -4.23 \text{ kN} \cdot \text{m}, \quad M_{BA} = -8.46 \text{ kN} \cdot \text{m}, \quad M_{BC} = 8.46 \text{ kN} \cdot \text{m}, \quad M_{CB} = -18.18 \text{ kN} \cdot \text{m}, \quad M_{CD} = 1.72 \text{ kN} \cdot \text{m}, \quad M_{CE} = 16.44 \text{ kN} \cdot \text{m} \]
\[ \theta_B = \frac{-5.29}{EI} \]

\[ \theta_C = \frac{2.86}{EI} \]

Shear diagram

Deflected curve

Moment diagram
**Example 8**

Determine the moments at each joint of the frame and draw the **quantitative bending moment** diagrams and **qualitative deflected curve**. The joints at $A$ and $D$ are fixed and joint $C$ is assumed pin-connected. $EI$ is constant for each member.
• Overview

- Unkowns

\[ \theta_B \text{ and } \Delta \]

- Boundary Conditions

\[ \theta_A = \theta_D = 0 \]

- Equilibrium Conditions

- Joint B

\[ \Sigma M_B = 0: \quad M_{BA} + M_{BC} = 0 \quad (1^*) \]

- Entire Frame

\[ \Sigma F_x = 0: \quad 10 - A_x - D_x = 0 \quad (2^*) \]
• Slope-Deflection Equations

\[
M_{AB} = \frac{2(EI)}{4} \theta_B + \frac{10(3)(1^2)}{4^2} + \frac{6EI\Delta}{4^2} + \frac{0.375EI\Delta}{4^2} \quad \text{--- (1)}
\]

\[
M_{BA} = \frac{4(EI)}{4} \theta_B - \frac{10(3^2)(1)}{4^2} + \frac{6EI\Delta}{4^2} + \frac{0.375EI\Delta}{4^2} \quad \text{--- (2)}
\]

\[
M_{BC} = \frac{3(EI)}{3} \theta_B \quad \text{--- (3)}
\]

\[
M_{DC} = 0.375EI\Delta - \frac{1}{2} 0.375EI\Delta = 0.1875EI\Delta \quad \text{--- (4)}
\]

\[
\sum M_B = 0:
A_x = \frac{(M_{AB} + M_{BA})}{4}
\quad A_x = 0.375EI\theta_B + 0.1875EI\Delta + 1.563 \quad \text{--- (5)}
\]

\[
\sum M_C = 0:
D_x = \frac{M_{DC}}{4} = 0.0468EI\Delta \quad \text{--- (6)}
\]
Equilibrium Conditions:

\[ M_{BA} + M_{BC} = 0 \quad ---(1^*) \]
\[ 10 - A_x - D_x = 0 \quad ---(2^*) \]

Slope-Deflection Equations:

\[ M_{AB} = \frac{2(EI)}{4} \theta_B + 5.625 + 0.375EI\Delta \quad ---(1) \]
\[ M_{BA} = \frac{4(EI)}{4} \theta_B - 5.625 + 0.375EI\Delta \quad ---(2) \]
\[ M_{BC} = \frac{3(EI)}{3} \theta_B \quad ---(3) \]
\[ M_{DC} = 0.1875EI\Delta \quad ---(4) \]

• Solve equation

Substitute (2) and (3) in (1*)
\[ 2EI \theta_B + 0.375EI \Delta = 5.625 \quad ----(7) \]

Substitute (5) and (6) in (2*)
\[ -0.375EI \theta_B - 0.235EI\Delta = -8.437 \quad ----(8) \]

From (7) and (8) can solve;
\[ \theta_B = \frac{-5.6}{EI} \quad \Delta = \frac{44.8}{EI} \]

Substitute \( \theta_B = \frac{-5.6}{EI} \) and \( \Delta = \frac{44.8}{EI} \) in (1) to (6)

\[ M_{AB} = 15.88 \text{ kN}\cdot\text{m} \]
\[ M_{BA} = 5.6 \text{ kN}\cdot\text{m} \]
\[ M_{BC} = -5.6 \text{ kN}\cdot\text{m} \]
\[ M_{DC} = 8.42 \text{ kN}\cdot\text{m} \]
\[ A_x = 7.9 \text{ kN} \]
\[ D_x = 2.1 \text{ kN} \]
$M_{AB} = 15.88 \text{kN} \cdot \text{m}$, $M_{BA} = 5.6 \text{kN} \cdot \text{m}$, $M_{BC} = -5.6 \text{kN} \cdot \text{m}$, $M_{DC} = 8.42 \text{kN} \cdot \text{m}$, $A_x = 7.9 \text{kN}$, $D_x = 2.1 \text{kN}$,
**Example 9**

From the frame shown use the slope-deflection method to:
(a) Determine the **end moments** of each member and **reactions** at supports
(b) Draw the **quantitative bending moment diagram**, and also draw the **qualitative deflected** shape of the entire frame.
• Overview

Boundary Conditions

Equilibrium Conditions

- Joint B

\[ \sum M_B = 0 : \quad M_{BA} + M_{BC} = 0 \quad \text{---(1*)} \]

- Entire Frame

\[ + \quad \sum F_x = 0 : \quad 10 - A_x - D_x = 0 \quad \text{---(2*)} \]
• Slope-Deflection Equation

\[ M_{AB} = \frac{2(EI)}{4} \theta_B + 0.375EI\Delta + 5 \quad - - - (1) \]

\[ M_{BC} = \frac{3(2EI)}{4} \theta_B - 0.2813EI\Delta \quad - - - (3) \]
• Horizontal reactions

\[ A_x = \frac{(M_{BA} + M_{AB} - 20)}{4} \quad ----(5) \]

\[ D_x = \frac{(M_{DC} - (3/4)M_{BC})}{4} \quad ----(6) \]
**Equilibrium Conditions:**

\[ M_{BA} + M_{BC} = 0 \quad \text{--- (1*)} \]
\[ 10 - A_x - D_x = 0 \quad \text{--- (2*)} \]

**Slope-Deflection Equation:**

\[ M_{AB} = \frac{2(EI)}{4} \theta_B + 5 + \frac{6EI\Delta}{4^2} \quad \text{--- (1)} \]
\[ M_{BA} = \frac{4(EI)}{4} \theta_B - 5 + \frac{6EI\Delta}{4^2} \quad \text{--- (2)} \]
\[ M_{BC} = \frac{3(2EI)}{4} \theta_B - \frac{3(2EI)(0.75\Delta)}{4^2} \quad \text{--- (3)} \]
\[ M_{DC} = \frac{3(2.5EI)(1.25\Delta)}{5^2} \quad \text{--- (4)} \]

**Horizontal reactions at supports:**

\[ A_x = \frac{(M_{BA} + M_{AB} - 20)}{4} \quad \text{--- (5)} \]
\[ D_x = \frac{M_{DC} - \frac{3}{4}M_{BC}}{4} \quad \text{--- (6)} \]

**• Solve equations**

Substitute (2) and (3) in (1*)

\[ 2.5EI\theta_B + 0.0938EI\Delta - 5 = 0 \quad \text{--- (7)} \]

Substitute (5) and (6) in (2*)

\[ 0.0938EI\theta_B + 0.334EI\Delta - 5 = 0 \quad \text{--- (8)} \]

From (7) and (8) can solve;

\[ \theta_B = \frac{1.45}{EI} \quad \Delta = \frac{-14.56}{EI} \]

Substitute \( \theta_B = \frac{1.45}{EI} \) and \( \Delta = \frac{-14.56}{EI} \) in (1) to (6)

\[ M_{AB} = 15.88 \text{ kN} \cdot \text{m} \]
\[ M_{BA} = 5.6 \text{ kN} \cdot \text{m} \]
\[ M_{BC} = -5.6 \text{ kN} \cdot \text{m} \]
\[ M_{DC} = 8.42 \text{ kN} \cdot \text{m} \]
\[ A_x = 7.9 \text{ kN} \]
\[ D_x = 2.1 \text{ kN} \]
\[ M_{AB} = 11.19 \, \text{kN}\cdot\text{m} \]
\[ M_{BA} = 1.91 \, \text{kN}\cdot\text{m} \]
\[ M_{BC} = -1.91 \, \text{kN}\cdot\text{m} \]
\[ M_{DC} = 5.46 \, \text{kN}\cdot\text{m} \]

\[ A_x = 8.28 \, \text{kN}\cdot\text{m} \]
\[ D_x = 1.72 \, \text{kN}\cdot\text{m} \]
Example 10

From the frame shown use the moment distribution method to:
(a) Determine all the reactions at supports, and also
(b) Draw its quantitative shear and bending moment diagrams, and qualitative deflected curve.

![Diagram of the frame with support reactions and moments labeled.]

- 20 kN/m load on the left side.
- Supports A and D are fixed.
- Support B is pinned.
- Support C is a pin.
- Labels for moments: 3EI at B, 2EI at C, 4EI at D.
• Overview

• Unknowns

\[ \theta_B \text{ and } \Delta \]

• Boundary Conditions

\[ \theta_A = \theta_D = 0 \]

• Equilibrium Conditions

- Joint B

\[ \Sigma M_B = 0 : \quad M_{BA} + M_{BC} = 0 \quad \text{--- (1*)} \]

- Entire Frame

\[ \Sigma F_x = 0 : \quad 60 - A_x - D_x = 0 \quad \text{--- (2*)} \]
• Slope-Deflection Equation

\[ M_{AB} = \frac{4(2EI)}{3} \theta_A + \frac{2(2EI)}{3} \theta_B + 15 + 1.333EI\Delta = 1.333EI\theta_B + 15 + 1.333EI\Delta \quad ---------\text{(1)} \]

\[ M_{BA} = \frac{2(2EI)}{3} \theta_A + \frac{4(2EI)}{3} \theta_B - 15 + 1.333EI\Delta = 2.667EI\theta_B - 15 + 1.333EI\Delta \quad ---------\text{(2)} \]

\[ M_{BC} = \frac{3(3EI)}{3} \theta_B = 3EI\theta_B \quad ---------\text{(3)} \]

\[ M_{DC} = \frac{3(4EI)}{4} \theta_D + 0.75EI\Delta = 0.75EI\Delta \quad ---------\text{(4)} \]
• Horizontal reactions

\[ \sum M_B = 0: \]
\[ A_x = \frac{M_{BA} + M_{AB} + 60(1.5)}{3} \]
\[ A_x = 1.333EI\theta_B + 0.889EI\Delta + 30 \quad ------(5) \]

\[ \sum M_C = 0: \]
\[ D_x = \frac{M_{DC}}{4} = 0.188EI\Delta \quad ------(6) \]
Equilibrium Conditions

\[ M_{BA} + M_{BC} = 0 \quad --- (1^*) \]
\[ 60 - A_x - D_x = 0 \quad --- (2^*) \]

Equation of moment

\[ M_{AB} = 1.333 EI \theta_B + 15 + 1.333 EI \Delta \quad --- (1) \]
\[ M_{BA} = 2.667 EI \theta_B - 15 + 1.333 EI \Delta \quad --- (2) \]
\[ M_{BC} = 3 EI \theta_B \quad --- (3) \]
\[ M_{DC} = 0.75 EI \Delta \quad --- (4) \]

Horizontal reaction at support

\[ A_x = 1.333 EI \theta_B + 0.889 EI \Delta + 30 \quad --- (5) \]
\[ D_x = 0.188 EI \Delta \quad --- (6) \]

\[ \theta_B = \frac{-5.51}{EI} \quad \Delta = \frac{34.67}{EI} \]

Substitute \( \theta_B \) and \( \Delta \) in (1)to (6)

\[ M_{AB} = 53.48 \text{ kN} \cdot \text{m} \]
\[ M_{BA} = 16.52 \text{ kN} \cdot \text{m} \]
\[ M_{BC} = -16.52 \text{ kN} \cdot \text{m} \]
\[ M_{DC} = 26.0 \text{ kN} \cdot \text{m} \]
\[ A_x = 53.48 \text{ kN} \]
\[ D_x = 6.52 \text{ kN} \]

\[ 5.667 EI \theta_B + 1.333 EI \Delta = 15 \quad --- (7) \]

\[ -1.333 EI \theta_B - 1.077 EI \Delta = -30 \quad --- (8) \]

From (7) and (8), solve equations;
\[ M_{AB} = 53.87 \text{ kN} \cdot \text{m} \]
\[ M_{BA} = 16.52 \text{ kN} \cdot \text{m} \]
\[ M_{BC} = -16.52 \text{ kN} \cdot \text{m} \]
\[ M_{DC} = 26.0 \text{ kN} \cdot \text{m} \]
\[ A_x = 53.48 \text{ kN} \]
\[ D_x = 6.52 \text{ kN} \]
Example

The statically indeterminate beam shown in the figure is to be analysed.

- Members AB, BC, CD have the same length $L = 10$ m.
- Flexural rigidities are $EI$, $2EI$, $EI$ respectively.
- Concentrated load of magnitude $P = 10$ kN acts at a distance $a = 3$ m from the support A.
- Uniform load of intensity $q = 1$ kN/m acts on BC.
- Member CD is loaded at its midspan with a concentrated load of magnitude $P = 10$ kN.

In the following calculations, clockwise moments and rotations are positive.

Degrees of freedom

Rotation angles $\theta_A$, $\theta_B$, $\theta_C$, $\theta_D$ of joints A, B, C, D respectively are taken as the unknowns. There are no chord rotations due to other causes including support settlement.

Fixed end moments

Fixed end moments are:

\[
M_{AB}^f = -\frac{Pab^2}{L^2} = -\frac{10 \times 3 \times 7^2}{10} = -14.7 \text{ kN m}
\]
\[
M_{BA}^f = \frac{Pa^2b}{L^2} = \frac{10 \times 3^2 \times 7}{10} = 6.3 \text{ kN m}
\]
\[
M_{BC}^f = -\frac{qL^3}{12} = -\frac{1 \times 10^2}{12} = -8.333 \text{ kN m}
\]
\[
M_{CB}^f = \frac{qL^3}{12} = \frac{1 \times 10^2}{12} = 8.333 \text{ kN m}
\]
\[
M_{CD}^f = -\frac{PL^2}{10} = -\frac{10 \times 10}{10} = -12.5 \text{ kN m}
\]
\[
M_{DC}^f = \frac{PL^2}{8} = \frac{10 \times 10^8}{8} = 12.5 \text{ kN m}
\]
Slope deflection equations

The slope deflection equations are constructed as follows:

\[
M_{AB} = \frac{EI}{L} (4\theta_A + 2\theta_B) = 0.4EI\theta_A + 0.2EI\theta_B
\]
\[
M_{BA} = \frac{EI}{2L} (2\theta_A + 4\theta_B) = 0.2EI\theta_A + 0.4EI\theta_B
\]
\[
M_{BC} = \frac{EI}{2L} (4\theta_B + 2\theta_C) = 0.8EI\theta_B + 0.4EI\theta_C
\]
\[
M_{CB} = \frac{EI}{2L} (2\theta_B + 4\theta_C) = 0.4EI\theta_B + 0.8EI\theta_C
\]
\[
M_{CD} = \frac{EI}{L} (4\theta_C) = 0.4EI\theta_C
\]
\[
M_{DC} = \frac{EI}{L} (2\theta_C) = 0.2EI\theta_C
\]

Joint equilibrium equations

Joints A, B, C should suffice the equilibrium condition. Therefore

\[
\Sigma M_A = M_{AB} + M_{BA} = 0.4EI\theta_A + 0.2EI\theta_B - 14.7 = 0
\]
\[
\Sigma M_B = M_{BA} + M_{BC} + M_{CB} = 0.2EI\theta_A + 1.2EI\theta_B + 0.4EI\theta_C - 2.033 = 0
\]
\[
\Sigma M_C = M_{CD} + M_{DC} + M_{CB} = 0.4EI\theta_B + 1.2EI\theta_C - 4.167 = 0
\]

Rotation angles

The rotation angles are calculated from simultaneous equations above.

\[
\theta_A = \frac{40.219}{E\frac{I}{L}}
\]
\[
\theta_B = \frac{-6.937}{E\frac{I}{L}}
\]
\[
\theta_C = \frac{5.785}{E\frac{I}{L}}
\]

Member end moments

Substitution of these values back into the slope deflection equations yields the member end moments (in kNm):

\[
M_{AB} = 0.4 \times 40.219 + 0.2 \times (-6.937) - 14.7 = 0
\]
\[
M_{BA} = 0.2 \times 40.219 + 0.4 \times (-6.937) + 6.3 = 11.57
\]
\[
M_{BC} = 0.8 \times (-6.937) + 0.4 \times 5.785 - 8.333 = -11.57
\]
\[
M_{CB} = 0.4 \times (-6.937) + 0.8 \times 5.785 + 8.333 = 10.19
\]
\[
M_{CD} = 0.4 \times 5.785 - 12.5 = -10.19
\]
\[
M_{DC} = 0.2 \times 5.785 + 12.5 = 13.66
\]
Example: Analyze the propped cantilever shown by using slope deflection method. Then draw Bending moment and shear force diagram.

Solution: End A is fixed hence $\theta_A = 0$

End B is Hinged hence $\theta_B \neq 0$

Assume both ends are fixed and therefore fixed end moments are

$$F_{AB} = -\frac{wL^2}{12}, \quad F_{BA} = +\frac{wL^2}{12}$$

The Slope deflection equations for final moment at each end are

$$M_{AB} = F_{AB} + \frac{2EI}{L} (2\theta_A + \theta_B)$$

$$= -\frac{wL^2}{12} + \frac{2EI}{L} \theta_B \quad \rightarrow (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L} (2\theta_B + \theta_A)$$

$$= \frac{wL^2}{12} + \frac{4EI}{L} \theta_B \quad \rightarrow (2)$$

In the above equations there is only one unknown $\theta_B$.

To solve we have boundary condition at B;

Since B is simply supported, the BM at B is zero

ie. $M_{BA}=0$.

$\therefore$ From equation $(2) M_{BA} = \frac{wL^2}{12} + \frac{4EI}{L} \theta_B = 0$

$\therefore EI \theta_B = -\frac{wL^3}{48}$ - ve sign indicates the rotation is anticlockwise

Substituting the value of $EI \theta_B$ in equation $(1)$ and $(2)$ we have end moments
\[ M_{AB} = -\frac{wL^2}{12} + 2 \left( -\frac{wL^2}{48} \right) = -\frac{wL^2}{8} \quad \text{ve sign indicates moment is anticlockwise} \]

\[ M_{BA} = + \frac{wL^2}{12} + 4 \left( -\frac{wL^3}{48} \right) = 0 \]

\( M_{BA} \) has to be zero, because it is hinged.

Now consider the free body diagram of the beam and find reactions using equations of equilibrium.

\[
\sum M_B = 0 \\
R_A \times L = M_{AB} + wL \times \frac{L}{2} \\
\quad = \frac{wL^2}{8} + wL \times \frac{L}{2} = \frac{5}{8} wL \\
\therefore \quad R_A = \frac{5}{8} wL \\
\sum V = 0 \\
R_A + R_B = wL \\
R_B = wL - R_A = wL - \frac{5}{8} wL \\
\quad = \frac{3}{8} wL
\]
Problem can be treated as

The bending moment diagram for the given problem is as below

The max BM occurs where SF=0. Consider SF equation at a distance of x from right support
\[ S_x = -\frac{3}{8} w L + w X = 0 \]

\[ \therefore X = \frac{3}{8} L \]

Hence the max BM occurs at \( \frac{3}{8} L \) from support B

\[ M_{\text{max}} = M_x = \frac{3}{8} w L \times \frac{3}{8} L - \frac{w}{2} \left( \frac{3}{8} L \right)^2 \]

\[ = \frac{9}{128} w L^2 \]

And point of contra flexure occurs where BM=0, Consider BM equation at a distance of \( x \) from right support.

\[ M_x = \frac{3}{8} w L X - \frac{w}{2} X^2 = 0 \]

\[ \therefore X = \frac{3}{4} L \]

For shear force diagram, consider SF equation from B

\[ S_x = +\frac{3}{8} w L - w X \]

\[ S_x = 0 = S_B = +\frac{3}{8} w L \]

\[ S_x = L = S_A = +\frac{5}{8} w L \]

**Example:** Analyze two span continuous beam ABC by slope deflection method. Then draw Bending moment & Shear force diagram. Take EI constant

![Diagram](image)

**Solution:** Fixed end moments are:
\[ F_{AB} = -\frac{W_{ab}^2}{L^2} = -\frac{100 \times 4 \times 2^2}{6^2} = -44.44 \text{KNM} \]
\[ F_{BA} = +\frac{W_{ba}^2}{L^2} = +\frac{100 \times 4^2 \times 2}{6^2} = +88.90 \text{KNM} \]
\[ F_{BC} = -\frac{wL^2}{12} = -\frac{20 \times 5^2}{12} = -41.67 \text{KNM} \]
\[ F_{CB} = +\frac{wL^2}{12} = +\frac{20 \times 5^2}{12} = 41.67 \text{KNM} \]

Since A is fixed \( \theta_A = 0, \theta_B \neq 0, \theta_C \neq 0 \),

Slope deflection equations are:

\[ M_{AB} = F_{AB} + \frac{2EI}{L} (2\theta_A + \theta_B) \]
\[ = -44.44 + \frac{2EI}{6} \theta_B \]
\[ = -44.44 + \frac{1}{3} EI \theta_B \quad \rightarrow (1) \]

\[ M_{BA} = F_{BA} + \frac{2EI}{L} (2\theta_B + \theta_A) \]
\[ = +88.89 + \frac{2EI \times 2\theta_B}{6} \]
\[ = 88.89 + \frac{2}{3} EI \theta_B \quad \rightarrow (2) \]

\[ M_{BC} = F_{BC} + \frac{2EI}{L} (2\theta_B + \theta_C) \]
\[ = -41.67 + \frac{2EI}{5} (2\theta_B + \theta_C) \]
\[ = -41.67 + \frac{4}{5} EI \theta_B + \frac{2}{5} EI \theta_C \quad \rightarrow (3) \]

\[ M_{CB} = F_{CB} + \frac{2EI}{L} (2\theta_C + \theta_B) \]
\[ = +41.67 + \frac{2EI}{5} (2\theta_C + \theta_B) \]
\[ = 41.67 + \frac{4EI}{5} \theta_C + \frac{2}{5} EI \theta_B \quad \rightarrow (4) \]

In all the above four equations there are only two unknown \( \theta_B \) and \( \theta_C \). And accordingly the boundary conditions are

\[ -M_{BA} - M_{BC} = 0 \]
\[ M_{BA} + M_{BC} = 0 \]
ii \( M_{CB} = 0 \) since C is end simply support.

Now \( M_{BA} + M_{BC} = 88.89 + \frac{2}{3} \text{EI} \theta_B - 41.67 + \frac{4}{5} \text{EI} \theta_B + \frac{2}{5} \text{EI} \theta_C \)

\[ = 47.22 + \frac{22}{15} \text{EI} \theta_B + \frac{2}{5} \text{EI} \theta_C = 0 \quad \cdots \cdots > (5) \]

\[ M_{CB} = 41.67 + \frac{2}{5} \text{EI} \theta_B + \frac{4}{5} \text{EI} \theta_C = 0 \quad \cdots \cdots > (6) \]

Solving simultaneous equations 5 & 6 we get

\( \text{EI} \theta_B = -20.83 \quad \text{Rotation anticlockwise.} \)

\( \text{EI} \theta_C = -41.67 \quad \text{Rotation anticlockwise.} \)

Substituting in the slope definition equations

\( M_{AB} = -44.44 + \frac{1}{3} (-20.83) = -51.38 \text{ KNm} \)

\( M_{BA} = + 88.89 + \frac{2}{3} (-20.83) = +75.00 \text{ KNm} \)

\( M_{BC} = -41.67 + \frac{4}{5} (-20.83) + \frac{2}{5} (-41.67) = -75.00 \text{ KNm} \)

\( M_{CB} = + 41.67 + \frac{2}{5} (-20.83) + \frac{4}{5} (-41.67) = 0 \)

Reactions: Consider the free body diagram of the beam.

Find reactions using equations of equilibrium.

Span AB: \( \sum M_A = 0 \quad R_B \times 6 = 100 \times 4 + 75 - 51.38 \)

\[ \therefore R_B = 70.60 \text{ KN} \]

\( \sum V = 0 \quad R_A + R_B = 100 \text{ KN} \)
\[ \therefore R_A = 100-70.60 = 29.40 \text{ KN} \]

**Span BC:** \[ \Sigma M_C = 0 \quad R_B \times 5 = 20 \times 5 \times \frac{5}{2} + 75 \]
\[ \therefore R_B = 65 \text{ KN} \]

\[ \Sigma V = 0 \quad R_B + R_C = 20 \times 5 = 100 \text{ KN} \]
\[ R_C = 100 - 65 = 35 \text{ KN} \]

Using these data BM and SF diagram can be drawn.

**Max BM:**

**Span AB:** Max BM in span AB occurs under point load and can be found geometrically
\[ M_{\text{max}} = 113.33 - 51.38 \left( \frac{75 - 51.38}{6} \right) \times 4 = 46.20 \text{ KNM} \]

**Span BC:** Max BM in span BC occurs where shear force is zero or changes its sign. Hence consider SF equation w.r.t C

\[ S_x = 35 - 20x = 0 \]

\[ : \quad x = \frac{35}{20} = 1.75 \text{ m} \]

Max BM occurs at 1.75 m from C

\[ : \quad M_{\text{max}} = 35 \times 1.75 - 20 \times \frac{1.75^2}{2} = 30.625 \text{ KNM} \]

**Example:** Analyze continuous beam ABCD by slope deflection method and then draw bending moment diagram. Take EI constant.

![Diagram of continuous beam ABCD](image)

**Solution:**

\[ \theta_A = 0, \quad \theta_B = 0, \quad \theta_C = 0 \]

**FEMS**

\[ F_{AB} = - \frac{W a^2 b}{L^2} = - \frac{100 \times 4 \times 2^2}{6^2} = - 44.44 \text{ KNM} \]

\[ F_{BA} = + \frac{W a^2 b}{L^2} = + \frac{100 \times 4^2 \times 2}{6^2} = + 88.88 \text{ KNM} \]

\[ F_{BC} = - \frac{w L^2}{12} = - \frac{20 \times 5^2}{12} = - 41.67 \text{ KNM} \]

\[ F_{CB} = + \frac{w L^2}{12} = + \frac{20 \times 5^2}{12} = + 41.67 \text{ KNM} \]

\[ F_{CD} = - 20 \times 1.5 = - 30 \text{ KNM} \]

**Slope deflection equations:**

\[ M_{AB} = F_{AB} + \frac{2EI}{L} (\theta_A + \theta_B) = -44.44 + \frac{1}{3} EI \theta_B \quad \text{----------> (1)} \]

\[ M_{BA} = F_{BA} + \frac{2EI}{L} (\theta_B + \theta_A) = +88.89 + \frac{2}{3} EI \theta_B \quad \text{----------> (2)} \]

\[ M_{BC} = F_{BC} + \frac{2EI}{L} (\theta_B + \theta_C) = -41.67 + \frac{4}{5} EI \theta_B + \frac{2}{5} EI \theta_C \quad \text{----------> (3)} \]

\[ M_{CB} = F_{CB} + \frac{2EI}{L} (\theta_C + \theta_B) = +41.67 + \frac{4}{5} EI \theta_C + \frac{2}{5} EI \theta_B \quad \text{----------> (4)} \]
\[ M_{CD} = -30 \text{ KNM} \]

In the above equations we have two unknown rotations \( \theta_B \) and \( \theta_C \), accordingly the boundary conditions are:

\[ M_{BA} + M_{BC} = 0 \]
\[ M_{CB} + M_{CD} = 0 \]

Now,
\[ M_{BA} + M_{BC} = 88.89 + \frac{2}{3} EI \theta_B - 41.67 + \frac{4}{5} EI \theta_B + \frac{2}{5} EI \theta_C \]
\[ = 47.22 + \frac{22}{15} EI \theta_B + \frac{2}{5} EI \theta_C = 0 \]
\[ \text{----------} > (5) \]

And,
\[ M_{CB} + M_{CD} = +41.67 + \frac{4}{5} EI \theta_C + \frac{2}{5} EI \theta_B - 30 \]
\[ = 11.67 + \frac{2}{5} EI \theta_B + \frac{4}{5} EI \theta_C \]
\[ \text{----------} > (6) \]

Solving (5) and (6) we get
\[ EI \theta_B = -32.67 \text{ Rotation @ B anticlockwise} \]
\[ EI \theta_C = +1.75 \text{ Rotation @ B clockwise} \]

Substituting value of \( EI \theta_B \) and \( EI \theta_C \) in slope deflection equations we have
\[ M_{AB} = -44.44 + \frac{1}{2} (-32.67) = -61.00 \text{ KNM} \]
\[ M_{BA} = +88.89 + \frac{2}{3} (-32.67) = +67.11 \text{ KNM} \]
\[ M_{BC} = -41.67 + \frac{4}{5} (-32.67) = + \frac{2}{5} (1.75) = -67.11 \text{ KNM} \]
\[ M_{CB} = +41.67 + \frac{4}{5} (1.75) + \frac{2}{5} (-32.67) = +30.00 \text{ KNM} \]
\[ M_{CD} = -30 \text{ KNM} \]

Reactions: Consider free body diagram of beam AB, BC and CD as shown
Span AB

\[ R_B \times 6 = 100 \times 4 + 67.11 - 61 \]
\[ R_B = 67.69 \text{ KN} \]
\[ R_A = 100 - R_B = 32.31 \text{ KN} \]

Span BC

\[ R_C \times 5 = 20 \times \frac{5}{2} \times 5 + 30 - 67.11 \]
\[ R_C = 42.58 \text{ KN} \]
\[ R_B = 20 \times 5 - R_B = 57.42 \text{ KN} \]

Maximum Bending Moments:

Span AB: Occurs under point load
Max = 133.3361 \left( \frac{67.11-61}{6} \times 4 \right) = 68.26\text{ KNM}

Span BC: where SF=0, consider SF equation with C as reference

\[ S_x = 42.58 - 20x = 0 \]
\[ x = \frac{42.58}{20} = 2.13\text{ m} \]

\[ M_{\text{max}} = 42.58 \times 2.13 - 20 \times \frac{2.13^2}{2} - 30 = 15.26\text{ KNM} \]

**Example:** Analyse the continuous beam ABCD shown in figure by slope deflection method. The support B sinks by 15mm.

Take \( E = 200 \times 10^5 \text{ KN/m}^2 \) and \( I = 120 \times 10^{-6} \text{ m}^4 \)

**Solution:**

In this problem \( \theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \delta = 15\text{ mm} \)

FEMs:

\[ F_{AB} = - \frac{W_{ab}L^2}{L^2} = -44.44\text{ KNM} \]
\[ F_{BA} = + \frac{W_{a}bL^2}{L^2} = +88.89\text{ KNM} \]
\[ F_{BC} = - \frac{wL^2}{8} = -41.67\text{ KNM} \]
\[ F_{CB} = + \frac{wL^2}{8} = +41.67\text{ KNM} \]

FEM due to yield of support B

For span AB:
\[ m_{ab} = m_{ba} = -\frac{6El}{L^2} \delta \]
\[ = -\frac{6 \times 200}{6^2} \times 10^5 \times 120 \times 10^{-6} \times \frac{15}{1000} = -6 \text{ KNM} \]

For span BC:
\[ m_{bc} = m_{cb} = +\frac{6El}{L^2} \delta \]
\[ = +\frac{6 \times 200}{5^2} \times 10^5 \times 120 \times 10^{-6} \times \frac{15}{1000} = +8.64 \text{ KNM} \]

Slope deflection equation
\[ M_{AB} = F_{AB} + \frac{2El}{L} (2\theta_A + \theta_B - \frac{3\delta}{L}) \]
\[ = F_{AB} + \frac{El}{L} (2\theta_A + \theta_B) - \frac{6El}{L^2} \delta \]
\[ = -44.44 + \frac{1}{3} El\theta_B - 6 \]
\[ = -50.44 + \frac{1}{3} El\theta_B \quad \text{---------> (1)} \]

\[ M_{BA} = F_{BA} + \frac{2El}{L} (2\theta_B + \theta_A) - \frac{6El}{L^2} \delta \]
\[ = +88.89 + \frac{2}{3} El\theta_B - 6 \]
\[ = +82.89 + \frac{2}{3} El\theta_B \quad \text{---------> (2)} \]

\[ M_{BC} = F_{BC} + \frac{2El}{L} (2\theta_B + \theta_C) + \frac{6El}{L^2} \delta \]
\[ = -41.67 + \frac{2}{5} El(2\theta_B + \theta_C) + 8.64 \]
\[ = -33.03 + \frac{4}{5} El\theta_B + \frac{2}{5} El\theta_C \quad \text{---------> (3)} \]

\[ M_{CB} = F_{CB} + \frac{2El}{L} (2\theta_C + \theta_B) + \frac{6El}{L^2} \delta \]
\[ = +41.67 + \frac{2}{5} El(2\theta_C + \theta_B) + 8.64 \]
\[ = +50.31 + \frac{4}{5} El\theta_C + \frac{2}{5} El\theta_B \quad \text{---------> (4)} \]

\[ M_{CD} = -30 \text{ KNM} \quad \text{---------> (5)} \]

There are only two unknown rotations \( \theta_B \) and \( \theta_C \). Accordingly the boundary conditions are
M_{BA} + M_{BC} = 0
M_{CB} + M_{CD} = 0

Now, \[ M_{BA} + M_{BC} = 49.86 + \frac{22}{15} \theta_{B} + \frac{2}{5} \theta_{C} = 0 \]
\[ M_{CB} + M_{CD} = 20.31 + \frac{2}{5} \theta_{B} + \frac{4}{5} \theta_{C} = 0 \]

Solving these equations we get
\[ \theta_{B} = -31.35 \text{ Anticlockwise} \]
\[ \theta_{C} = -9.71 \text{ Anticlockwise} \]

Substituting these values in slope deflections we get the final moments:
\[ M_{AB} = -50.44 + \frac{1}{3} (-31.35) = -60.89 \text{ KNM} \]
\[ M_{BA} = +82.89 + \frac{2}{3} (-31.35) = +61.99 \text{ KNM} \]
\[ M_{BC} = -33.03 + \frac{4}{5} (-31.35) + \frac{2}{5} (-9.71) = -61.99 \text{ KNM} \]
\[ M_{CB} = +50.31 + \frac{4}{5} (-9.71) + \frac{2}{5} (-31.35) = +30.00 \text{ KNM} \]
\[ M_{CD} = -30 \text{ KNM} \]

Consider the free body diagram of continuous beam for finding reactions

Reactions:

Span AB:
\[ R_{B} \times 6 = 100 \times 4 + 61.99 - 60.89 \]
\[ R_{B} = 66.85 \]
\[ R_{A} = 100 - R_{B} \]
\[ = 33.15 \text{ KN} \]

Span BC:
\[ R_{B} \times 5 = 20 \times 5 \times \frac{5}{2} + 61.99 - 30 \]
\[ R_{B} = 56.40 \text{ KN} \]
\[ R_{C} = 20 \times 5 - R_{B} \]
Example: Three span continuous beam ABCD is fixed at A and continuous over B, C and D. The beam subjected to loads as shown. Analyse the beam by slope deflection method and draw bending moment and shear force diagram.

Solution:

Since end A is fixed \( \theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D = 0 \)

FEMs:

\[
F_{AB} = \frac{-WI}{8} = \frac{-60 \times 4}{8} = -30 \text{ KNM}
\]

\[
F_{BA} = \frac{WI}{8} = \frac{60 \times 4}{8} = +30 \text{ KNM}
\]

\[
F_{BC} = \frac{M}{4} = +12.5 \text{ KNM}
\]

\[
F_{CB} = \frac{M}{4} = +12.5 \text{ KNM}
\]
\[ F_{CD} = - \frac{wL}{12} = -10 \times 4^2 = -13.33 \text{ KNM} \]
\[ F_{DC} = + \frac{wL}{12} = +10 \times 4^2 = +13.33 \text{ KNM} \]

Slope deflection equations:

\[ M_{AB} = \frac{F_{AB} + 2EI}{L} (2\theta_A + \theta_B) \]
\[ = -30 + \frac{2EI}{4} (0 + \theta_B) \]
\[ = -30 + 0.5E\theta_B \]
\[ \text{-------------} > (1) \]

\[ M_{BA} = \frac{F_{BA} + 2EI}{L} (2\theta_B + \theta_A) \]
\[ = 30 + \frac{2EI}{4} (2\theta_B + 0) \]
\[ = +30 + E\theta_B \]
\[ \text{-------------} > (2) \]

\[ M_{BC} = \frac{F_{BC} + 2EI}{L} (2\theta_B + \theta_C) \]
\[ = 12.5 + \frac{2EI}{4} (2\theta_B + \theta_C) \]
\[ = 12.5 + E\theta_B + 0.5E\theta_C \]
\[ \text{-------------} > (3) \]

\[ M_{CB} = \frac{F_{CB} + 2EI}{L} (2\theta_C + \theta_B) \]
\[ = 12.5 + \frac{2EI}{4} (2\theta_C + \theta_B) \]
\[ = 12.5 + E\theta_C + 0.5E\theta_B \]
\[ \text{-------------} > (4) \]

\[ M_{CD} = \frac{F_{CD} + 2EI}{L} (2\theta_C + \theta_D) \]
\[ = -13.33 + \frac{2EI}{4} (2\theta_C + \theta_D) \]
\[ = -13.33 + E\theta_C + 0.5E\theta_D \]
\[ \text{-------------} > (5) \]

\[ M_{DC} = \frac{F_{DC} + 2EI}{L} (2\theta_D + \theta_C) \]
\[ = 13.33 + \frac{2EI}{4} (2\theta_D + \theta_C) \]
\[ = 13.33 + 0.5E\theta_C + E\theta_D \]
\[ \text{-------------} > (6) \]

In the above Equations there are three unknowns, \(E\theta_B, E\theta_C\) & \(E\theta_D\), accordingly the boundary conditions are:

i. \(M_{BA} + M_{BC} = 0\)
ii. \(M_{CB} + M_{CD} = 0\)
iii. \(M_{DC} = 0\) (\(\therefore\) hinged)

Now
By solving (7), (8) & (9), we get

\[
\begin{align*}
\theta_B &= -24.04 \\
\theta_C &= +11.15 \\
\theta_D &= -18.90
\end{align*}
\]

By substituting the values of \( \theta_B, \theta_C \) and \( \theta_D \) in respective equations we get

\[
\begin{align*}
M_{AB} &= -30 + 0.5(-24.04) = -42.02 \text{ KNM} \\
M_{BA} &= +30 + (-24.04) = +5.96 \text{ KNM} \\
M_{BC} &= +12.5 + (-24.04) + 0.5(+11.15) = -5.96 \text{ KNM} \\
M_{CB} &= +12.5 + 11.15 + 0.5(-24.04) = +11.63 \text{ KNM} \\
M_{CD} &= -13.33 + 11.15 + 0.5(-18.90) = -11.63 \text{ KNM} \\
M_{DC} &= +13.33 + 0.5(11.15) + (-18.90) = 0 \text{ KNM}
\end{align*}
\]

Reactions: Consider the free body diagram of beam.

**Beam AB:**

\[
R_B = \frac{60 \times 2 + 5.96 - 42.02}{4} = 20.985 \text{ KN}
\]

\[\therefore R_A = 60 - R_B = 30.015 \text{ KN}\]

**Beam BC:**

\[
R_C = \frac{11.63 + 50 - 5.96}{4} = 13.92 \text{ KN}
\]

\[\therefore R_B = -R_C = -13.92 \text{ KN} \quad \therefore R_B \text{ is dow nw arc}\]

**Beam CD:**
\[
R_D = \frac{10 \times 4 \times 2 - 11.63}{4} = 17.09 \text{ KN}
\]
\[
\therefore R_C = 10 \times 4 - R_D = 22.91 \text{ KN}
\]

**Example:** Analyse the continuous beam shown using slope deflection method. Then draw bending moment and shear force diagram.

**Solution:** In this problem \( \theta_A = 0 \), \( \therefore \) end \( A \) is fixed

FEMs:
\[
F_{AB} = - \frac{wL^2}{12} = - \frac{10 \times 8^2}{12} = - 53.33 \text{ KNM}
\]
\[ F_{BA} = \frac{w l_f}{12} = +53.33 \text{ KNM} \]
\[ F_{BC} = -\frac{Wl}{8} = -\frac{30 \times 6}{8} = -22.50 \text{ KNM} \]
\[ F_{CD} = +\frac{WL}{8} = +22.50 \text{ KNM} \]

Slope deflection equations:

\[ M_{AB} = F_{AB} + \frac{2EI}{L} (2\theta_A + \theta_B) \]
\[ = -53.33 + \frac{2E \times 3I}{8} (0 + \theta_B) \]
\[ = -53.33 + \frac{3}{4} EI \theta_B \]
\[ \text{----------- \text{> (1)}} \]

\[ M_{BA} = F_{BA} + \frac{2EI}{L} (2\theta_B + \theta_A) \]
\[ = +53.33 + \frac{2E \times 3I}{8} (2\theta_B + 0) \]
\[ = 53.33 + \frac{3}{2} EI \theta_B \]
\[ \text{----------- \text{> (2)}} \]

\[ M_{BC} = F_{BC} + \frac{2EI}{L} (2\theta_B + \theta_C) \]
\[ = -22.5 + \frac{2E \times 3I}{6} (2\theta_B + \theta_C) \]
\[ = -22.5 + \frac{4}{3} EI \theta_B + \frac{2}{3} EI \theta_C \]
\[ \text{----------- \text{> (3)}} \]

\[ M_{CB} = F_{CB} + \frac{2EI}{L} (2\theta_C + \theta_B) \]
\[ = +22.5 + \frac{2E \times 3I}{6} (2\theta_C + \theta_B) \]
\[ = +22.5 + \frac{4}{3} EI \theta_C + \frac{2}{3} EI \theta_B \]
\[ \text{----------- \text{> (4)}} \]

In the above equation there are two unknown \( \theta_B \) and \( \theta_C \), accordingly the boundary conditions are:

i \( -M_{BA} - M_{BC} - 24 = 0 \)

ii \( M_{CB} = 0 \)

Now, \( M_{BA} + M_{BC} - 24 = 53.33 + \frac{3}{2} EI \theta_B - 22.5 + \frac{4}{3} EI \theta_B + \frac{2}{3} EI \theta_C + 24 \]
\[ = +54.83 + \frac{17}{6} EI \theta_B + \frac{2}{3} EI \theta_C = 0 \] \( \text{----------- \text{> (5)}} \)
and \( M_{CB} = 22.5 + \frac{4}{3} \theta_c + \frac{2}{3} \theta_b = 0 \)

\[ \therefore \frac{2}{3} \theta_c = -11.25 - \frac{1}{3} \theta_b \]

\[ \therefore \theta_c = -8.159 \text{ rotation anticlockwise} \]

Substituting in eqn. (5)

\[ 54.83 + \frac{17}{6} \theta_b - 11.25 - \frac{1}{3} \theta_b = 0 \]

\[ + 44.58 + \frac{15}{6} \theta_b = 0 \]

\[ \therefore \theta_b = -\frac{44.58 \times 6}{15} = -17.432 \text{ rotation anticlockwise} \]

\[ \therefore \theta_c = \frac{3}{2} \left[ -11.25 - \frac{1}{3} (-17.432) \right] \]

Substituting \( \theta_b = -17.432 \) and \( \theta_c = -8.159 \) in the slope deflection equation we get

Final Moments:

\[ M_{AB} = -53.33 + \frac{3}{4} (-17.432) = -66.40 \text{ KN} \]

\[ M_{BA} = +53.33 + \frac{3}{2} (-17.432) = +27.18 \text{ KN} \]

\[ M_{BC} = -22.5 + \frac{4}{3} (-17.432) + \frac{2}{3} (-8.159) = -51.18 \text{ KN} \]

\[ M_{CB} = +22.5 + \frac{4}{3} (-8.159) + \frac{2}{3} (-17.432) = 0.00 \]

Reactions: Consider free body diagram of beams as shown

Span AB:

\[ R_B = \frac{27.18 - 66.40 + 10 \times 8 \times 4}{8} = 35.13 \text{ KN} \]

\[ \therefore R_A = 10 \times 8 - R_B = 44.87 \text{ KN} \]
Span BC:
\[ R_B = \frac{51.18 + 30 \times 3}{6} = 23.53 \text{ KN} \]
\[ R_C = 30 - R_B = 6.47 \text{ KN} \]

Max BM

Span AB: Max BM occurs where SF=0, consider SF equation with A as origin
\[ S_x = 44.87 - 10x = 0 \]
\[ x = 4.487 \text{ m} \]
\[ \therefore M_{max} = 44.87 \times 4.487 - 10 \times \frac{4.487^2}{2} - 64 = 36.67 \text{ KNM} \]

Span BC: Max BM occurs under point load
\[ BC \ M_{max} = 45 - \frac{51.18}{2} = 19.41 \text{ KNM} \]

Example: Analyse the beam shown in figure. End support C is subjected to an anticlockwise moment of 12 KNM.
Solution: In this problem $\theta_A = 0$, :: end is fixed

FEMs:

$$F_{BC} = -\frac{w l^3}{12} = -\frac{20 \times 4^2}{12} = -26.67 \text{ KNM}$$
$$F_{CB} = +\frac{w l^3}{12} = + 26.67 \text{ KNM}$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L} (2\theta_A + \theta_B)$$
$$= 0 + \frac{2EI}{L} (0 + \theta_B)$$
$$= EI \theta_B \quad \text{---------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L} (2\theta_B + \theta_A)$$
$$= 0 + \frac{2EI}{L} (2\theta_B + 0)$$
$$= 2EI \theta_B \quad \text{---------} > (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L} (2\theta_B + \theta_C)$$
$$= -26.67 + \frac{2EI \times 1.5L}{4} (2\theta_B + \theta_C)$$
$$= -26.67 + \frac{3EI \theta_B}{2} + \frac{3EI \theta_C}{4} \quad \text{---------} > (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L} (2\theta_C + \theta_B)$$
$$= +26.67 + \frac{2EI \times 1.5L}{4} (2\theta_C + \theta_B)$$
$$= +26.67 + \frac{3EI \theta_C}{2} + \frac{3EI \theta_B}{4} \quad \text{---------} > (4)$$

In the above equation there are two unknowns $\theta_B$ and $\theta_C$, accordingly the boundary conditions are

$$M_{BA} + M_{BC} = 0$$
$$M_{CB} + 12 = 0$$

Now, $M_{BA} + M_{BC} = 2EI \theta_B - 26.67 + \frac{3}{2} EI \theta_B + \frac{3}{4} EI \theta_C$

$$= \frac{7}{2} EI \theta_B + \frac{3}{4} EI \theta_C - 26.67 = 0 \quad \text{---------} > (5)$$
and \( M_{CB} + 12 = 26.67 + \frac{3}{2} \theta_c + \frac{3}{4} \theta_b + 12 \)
\[ = 38.67 + \frac{3}{4} \theta_b + \frac{3}{2} \theta_c = 0 \quad \cdots \quad (6) \]

From (5) and (6)
\[ \frac{7}{2} \theta_b + \frac{3}{4} \theta_c - 26.67 = 0 \]
\[ \frac{3}{8} \theta_b + \frac{3}{4} \theta_c + 19.33 = 0 \]
\[ \frac{25}{8} \theta_b - 46 = 0 \]
\[ \theta_b = +46 \times \frac{8}{25} = +14.72 \]

From (6)
\[ \theta_c = -\frac{2}{3} \left( 38.67 + \frac{3}{4} (14.72) \right) \]
\[ = -33.14 \quad \text{-ve sign indicates rotation anticlockwise} \]

Substituting \( \theta_b \) and \( \theta_c \) is slope deflection equations

\[ M_{AB} = \theta_b = +14.72 \quad \text{KNM} \]
\[ M_{BA} = 2\theta_b = 2(14.72) = 29.42 \quad \text{KNM} \]
\[ M_{BC} = -26.67 + \frac{3}{2} (14.72) + \frac{3}{4} (-33.14) = 29.44 \quad \text{KNM} \]
\[ M_{CB} = +26.67 + \frac{3}{2} (-33.14) + \frac{3}{4} (14.72) = -12 \quad \text{KNM} \]

Reaction: Consider free body diagrams of beam

Span AB:
\[ R_B = \frac{14.72 + 29.44}{4} = 11.04 \text{ KN} \]
\[ R_A = -R_B = -11.04 \text{ KN} \]

Span BC:
\[ R_B = \frac{29.44 + 12 + 20 \times 4 \times 2}{4} = 50.36 \text{ KN} \]
\[ R_C = 20 \times 4 - R_B = 29.64 \text{ KN} \]

Example: Analyse the simple frame shown in figure. End A is fixed and ends B & C are hinged. Draw the bending moment diagram.
Solution:

In this problem \( \theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D \neq 0, \)

FEMS:-

\[
F_{AB} = -\frac{W_{ab}^2}{L^2} = -\frac{120 \times 2 \times 4^2}{6^2} = -106.67 \text{ KNM}
\]

\[
F_{BA} = +\frac{W_{a}^2b}{L^2} = +\frac{120 \times 2^2 \times 4}{6^2} = +53.33 \text{ KNM}
\]

\[
F_{BC} = -\frac{Wf}{12} = -\frac{20 \times 4^2}{12} = -26.67 \text{ KNM}
\]

\[
F_{CB} = +\frac{Wf}{12} = +\frac{20 \times 4^2}{12} = +26.67 \text{ KNM}
\]

\[
F_{CD} = +\frac{WL}{8} = +\frac{20 \times 4}{8} = +10 \text{ KNM}
\]

\[
F_{DB} = -\frac{WL}{8} = -10 \text{ KNM}
\]

Slope deflections are

\[
M_{AB} = F_{AB} + \frac{2EI}{L} (2\theta_A + \theta_B)
\]

\[
= -106.67 + \frac{2EI}{6} (\theta_B) = -106.67 + \frac{2}{3} EI \theta_B
\]

\[\text{----- > (1)}\]

\[
M_{BA} = F_{BA} + \frac{2EI}{L} (2\theta_B + \theta_A)
\]

\[
= +53.33 + \frac{2EI}{6} (\theta_B) = +53.33 + \frac{4}{3} EI \theta_B
\]

\[\text{----- > (2)}\]

\[
M_{BC} = F_{CB} + \frac{2EI}{L} (2\theta_B + \theta_C)
\]

\[
= -26.67 + \frac{2E}{4} \times \frac{3I}{2} (2\theta_B + \theta_C) = -26.67 + \frac{3}{2} EI \theta_B + \frac{3}{4} EI \theta_C
\]

\[\text{----- > (3)}\]

\[
M_{CB} = F_{CB} + \frac{2EI}{L} (2\theta_C + \theta_B)
\]

\[
= +26.67 + \frac{2E}{4} \times \frac{3I}{2} (2\theta_C + \theta_B) = +26.67 + \frac{3}{2} EI \theta_C + \frac{3}{4} EI \theta_B
\]

\[\text{----- > (4)}\]

\[
M_{BD} = F_{BD} + \frac{2EI}{L} (2\theta_B + \theta_D)
\]

\[
= +10 + \frac{2EI}{4} (2\theta_B + \theta_D) = +10 + EI \theta_B + \frac{1}{2} EI \theta_D
\]

\[\text{----- > (5)}\]

\[
M_{DB} = F_{DB} + \frac{2EI}{L} (2\theta_D + \theta_B)
\]

\[
= -10 + \frac{2EI}{4} (2\theta_D + \theta_B) = -10 + EI \theta_D + \frac{1}{2} EI \theta_B
\]

\[\text{----- > (6)}\]

In the above equations we have three unknown rotations \( \theta_B, \theta_C, \theta_D \) accordingly we have three boundary conditions.

\[
M_{BA} + M_{BC} + M_{BD} = 0
\]
\[ M_{CB} = 0 \quad \text{Since C and D are hinged} \]
\[ M_{DB} = 0 \]

Now
\[ M_{BA} + M_{BC} + M_{BD} = 53.33 + \frac{4}{3} \theta_B - 26.67 + \frac{3}{2} \theta_B + \frac{3}{4} \theta_C + 10 + \theta_B + \frac{1}{2} \theta_D \]
\[ = 36.66 + \frac{23}{6} \theta_B + \frac{3}{4} \theta_C + \frac{1}{2} \theta_D = 0 \quad ----> (7) \]

\[ M_{CB} = 26.67 + \frac{3}{4} \theta_B + \frac{3}{2} \theta_C = 0 \quad ----> (8) \]

\[ M_{DB} = -10 + \frac{1}{2} \theta_B + \theta_D = 0 \quad ----> (9) \]

Solving equations 7, 8, & 9 we get
\[ \theta_B = -8.83 \]
\[ \theta_C = -13.36 \]
\[ \theta_D = +14.414 \]

Substituting these values in slope equations
\[ M_{AB} = -106.67 + \frac{2}{3} (-8.83) = -112.56 \text{ KNM} \]
\[ M_{BA} = 53.33 + \frac{4}{3} (-8.83) = 41.56 \text{ KNM} \]
\[ M_{BC} = -26.67 + \frac{3}{2} (-8.3) + \frac{3}{4} (-13.36) = -49.94 \text{ KNM} \]
\[ M_{CB} = +26.67 + \frac{3}{2} (-13.36) + \frac{3}{4} (-8.83) = 0 \]
\[ M_{BD} = 10 + (-8.83) + \frac{1}{2} (+14.414) = 8.38 \text{ KNM} \]
\[ M_{DB} = -10 + (14.414) + \frac{1}{2} (-8.83) = 0 \]
Reactions: Consider free body diagram of each members

Span AB:

\[ R_B = \frac{41.56 - 112.56 + 120 \times 2}{6} = 28.17 \text{ KN} \]

\[ \therefore R_A = 120 - R_B = 91.83 \text{ KN} \]

Span BC:

\[ R_B = \frac{49.94 + 20 \times 4 \times 2}{4} = 52.485 \text{ KN} \]

\[ \therefore R_C = 20 \times 4 - R_B = 27.515 \text{ KN} \]

Column BD:

\[ H_D = \frac{20 \times 2 - 8.33}{4} = 7.92 \text{ KN} \]

\[ \therefore H_B = 12.78 \text{ KN} \quad \left[ \therefore H_A + H_D = 20 \right] \]
Example: Analyse the portal frame shown in figure and also drawn bending moment and shear force diagram

Solution:

Symmetrical problem
- Sym frame + Sym loading
\( \theta_A = 0, \ \theta_B \neq 0, \ \theta_C \neq 0, \ \theta_D = 0 \)

FEMS

\[
F_{BC} = - \frac{W_1 ab^2}{L^2} - \frac{W_2 cd^2}{L^2} = \frac{80 \times 2 \times 4^2}{6^2} - \frac{80 \times 4 \times 2^2}{6^2} = -106.67 \text{ KNM}
\]

\[
F_{CB} = + \frac{W a^2 b}{L^2} + \frac{W c^2 d}{L^2} = +106.67 \text{ KNM}
\]

Slope deflection equations:
\[ M_{AB} = F_{AB} + \frac{2EI}{L} (2\theta_A + \theta_B) = 0 + \frac{2EI}{4} (0 + \theta_B) = \frac{1}{2} EI \theta_B \]  \quad \ldots \ldots (1)

\[ M_{BA} = F_{BA} + \frac{2EI}{L} (2\theta_B + \theta_A) = 0 + \frac{2EI}{4} (2\theta_B + 0) = EI \theta_B \]  \quad \ldots \ldots (2)

\[ M_{BC} = F_{BC} + \frac{2EI}{L} (2\theta_B + \theta_C) \]
\[ = -106.67 + \frac{2EI}{6} (2\theta_B + \theta_C) = -106.67 + \frac{4}{3} EI \theta_B + \frac{2}{3} EI \theta_C \]  \quad \ldots \ldots (3)

\[ M_{CB} = F_{CB} + \frac{2EI}{L} (2\theta_C + \theta_B) \]
\[ = +106.67 + \frac{2EI}{6} (2\theta_C + \theta_B) = +106.67 + \frac{4}{3} EI \theta_C + \frac{2}{3} EI \theta_B \]  \quad \ldots \ldots (4)

\[ M_{CD} = F_{CD} + \frac{2EI}{L} (2\theta_C + \theta_D) \]
\[ = 0 + \frac{2EI}{4} (2\theta_C + 0) = EI \theta_C \]  \quad \ldots \ldots (5)

\[ M_{DC} = F_{DC} + \frac{2EI}{L} (2\theta_D + \theta_C) \]
\[ = 0 + \frac{2EI}{4} (0 + \theta_C) = \frac{1}{2} EI \theta_C \]  \quad \ldots \ldots (6)

In the above equation there are two unknown rotations. Accordingly the boundary conditions are

\[ M_{BA} + M_{BC} = 0 \]
\[ M_{CB} + M_{CD} = 0 \]

Now \[ M_{BA} + M_{BC} = -106.67 + \frac{7}{3} EI \theta_B + \frac{2}{3} EI \theta_C = 0 \]  \quad \ldots \ldots (7)

\[ M_{CB} + M_{CD} = +106.67 + \frac{2}{3} EI \theta_B + \frac{7}{3} EI \theta_C = 0 \]  \quad \ldots \ldots (8)

Multiply by (7) and (8) by 2

\[ \begin{cases} -746.69 + \frac{49}{3} EI \theta_B + \frac{14}{3} EI \theta_C = 0 \\ +213.34 + \frac{4}{3} EI \theta_B + \frac{14}{3} EI \theta_C = 0 \end{cases} \]

\[ -960.03 + \frac{45}{3} EI \theta_B = 0 \]

\[ EI \theta_B = +960.03 \times \frac{3}{45} = + 64 \quad \text{Clockwise} \]

Using equation (7)
\[ \text{EI} \theta_C = -\frac{3}{2} \left[ -106.67 + \frac{7}{3} \text{EI} \theta_B \right] \]
\[ = -\frac{3}{2} \left[ -106.67 + \frac{7}{3} \times 64 \right] = -64 \text{ Anticlockwise} \]

Here we find \( \theta_B = -\theta_C \). It is obvious because the problem is symmetrical.

\[ \therefore \text{Final moments are} \]

\[ M_{AB} = +\frac{64}{2} = +32 \text{ KNM} \]
\[ M_{BA} = 64 \text{ KNM} \]
\[ M_{BC} = -106.67 + \frac{4}{3} 64 + \frac{2}{3}(-64) = -64 \text{ KNM} \]
\[ M_{CB} = +106.67 + \frac{4}{3}(-64) + \frac{2}{3}(64) = +64 \text{ KNM} \]
\[ M_{CD} = -64 \text{ KNM} \]
\[ M_{DC} = -\frac{1}{2} 64 = -32 \text{ KNM} \]

Consider free body diagram’s of beam and columns as shown.
By symmetrical we can write
\[ R_A = R_B = 60 \text{ KNM} \]
\[ R_D = R_C = 80 \text{ KNM} \]

Now consider free body diagram of column AB

Apply
\[ \sum M_B = 0 \]
\[ H_A \times 4 = 64 + 32 \]
\[ \therefore H_A = 24 \text{ KN} \]

Similarly from free body diagram of column CD

Apply
\[ \sum M_C = 0 \]
\[ H_A \times 4 = 64 + 32 \]
\[ \therefore H_D = 24 \text{ KN} \]

Check:
\[ \sum H = 0 \]
\[ H_A + H_D = 0 \]
Hence okay
Note: Since symmetrical, only half frame may be analysed. Using first three equations and taking $\theta_B = -\theta_C$

**Example:** Analyse the portal frame and then draw the bending moment diagram

**Solution:**
This is a symmetrical frame and unsymmetrically loaded, thus it is an unsymmetrical problem and there is a sway

Assume sway to right.

Here \( \theta_A = 0, \theta_D = 0, \theta_B \neq 0, \theta_D = 0 \)

**FEMS:**

\[
F_{BC} = -\frac{W a b^2}{L^2} = -\frac{80 \times 5 \times 3^2}{8^2} = -56.25 \text{ KNM}
\]

\[
F_{CB} = +\frac{W a^2 b^3}{L^2} = +\frac{80 \times 5^2 \times 3}{8^2} = +93.75 \text{ KNM}
\]

Slope deflection equations

\[
M_{AB} = F_{AB} + \frac{2EI}{L} \left( \theta_A - \frac{3\delta}{L} \right)
\]

\[
= 0 + \frac{2EI}{4} \left( \theta_A - \frac{3\delta}{4} \right) = \frac{1}{2} \theta_A - \frac{3}{8} \delta
\]

\[
\text{-------------------} > (1)
\]

\[
M_{BA} = F_{BA} + \frac{2EI}{L} \left( \theta_B - \frac{3\delta}{L} \right)
\]

\[
= 0 + \frac{2EI}{4} \left( \theta_B - \frac{3\delta}{4} \right) = \theta_B - \frac{3}{8} \delta
\]

\[
\text{-------------------} > (2)
\]

\[
M_{BC} = F_{BC} + \frac{2EI}{L} \left( \theta_B + \theta_C \right)
\]

\[
= -56.25 + \frac{2EI}{8} \left( \theta_B + \theta_C \right) = -56.25 + \frac{1}{2} \theta_B + \frac{1}{4} \theta_C
\]

\[
\text{-------------------} > (3)
\]

\[
M_{CB} = F_{CB} + \frac{2EI}{L} \left( \theta_C + \theta_B \right)
\]

\[
= +93.75 + \frac{2EI}{8} \left( \theta_C + \theta_B \right) = 93.75 + \frac{1}{2} \theta_C + \frac{1}{4} \theta_B
\]

\[
\text{-------------------} > (4)
\]

\[
M_{CD} = F_{CD} + \frac{2EI}{L} \left( \theta_C + \theta_D - \frac{3\delta}{L} \right)
\]

\[
= 0 + \frac{2EI}{4} \left( \theta_C + \theta_D - \frac{3\delta}{4} \right) = \theta_C - \frac{3}{8} \delta
\]

\[
\text{-------------------} > (5)
\]

\[
M_{DC} = F_{DC} + \frac{2EI}{L} \left( \theta_D + \theta_C - \frac{3\delta}{L} \right)
\]

\[
= 0 + \frac{2EI}{4} \left( \theta_D + \theta_C - \frac{3\delta}{4} \right) = \frac{1}{2} \theta_D - \frac{3}{8} \delta
\]

\[
\text{-------------------} > (6)
\]

In the above equation there are three unknowns \( \theta_B, \theta_C \) and \( \delta \), accordingly the boundary conditions are,
\[ M_{BA} + M_{BC} = 0 \quad \text{-- -- > Joint conditions} \]
\[ M_{CB} + M_{CD} = 0 \]
\[ H_B + H_D + \Sigma P = 0 \quad \text{-- -- > Shear condition} \]
\[ \text{i.e.,} \quad \frac{M_{AB} + M_{BA}}{4} + \frac{M_{CD} + M_{DC}}{4} = 0 \]
\[ \therefore \quad M_{AB} + M_{BA} + M_{CD} + M_{DC} = 0 \]

Now, \[ M_{BA} + M_{BC} = EI \theta_B - \frac{3}{8} EI \delta - 56.25 \frac{1}{2} EI \theta_B + \frac{1}{4} EI \theta_C = 0 \]
\[ = -56.25 + \frac{3}{2} EI \theta_B + \frac{1}{4} EI \theta_C - \frac{3}{8} EI \delta = 0 \quad \text{-- -- -- -- > (7)} \]

And \[ M_{CB} + M_{CD} = 93.75 \frac{1}{2} EI \theta_C + \frac{1}{4} EI \theta_B + EI \theta_C - \frac{3}{8} EI \delta = 0 \]
\[ = 93.75 + \frac{3}{2} EI \theta_B + \frac{1}{4} EI \theta_C - \frac{3}{8} EI \delta = 0 \quad \text{-- -- -- -- > (8)} \]

And \[ M_{AB} + M_{BA} + M_{CD} + M_{DC} = \frac{1}{2} EI \theta_B - \frac{3}{8} EI \delta + EI \theta_B - \frac{3}{8} EI \delta + EI \theta_C - \frac{3}{8} EI \delta \]
\[ + \frac{1}{2} EI \theta_C - \frac{3}{8} EI \delta \]
\[ = \frac{3}{2} EI \theta_B + \frac{3}{2} EI \theta_C - \frac{3}{2} EI \delta = 0 \quad \text{-- -- -- -- > (9)} \]

From (9) \[ EI \delta = EI \theta_B + EI \theta_C \]

Substitute in (7) & (8)

Eqn (7)
\[ -56.25 + \frac{3}{2} EI \theta_B + \frac{1}{4} EI \theta_C - \frac{3}{8} [EI \theta_B + EI \theta_C] = 0 \]
\[ -56.25 + \frac{9}{8} EI \theta_B - \frac{1}{8} EI \theta_C = 0 \quad \text{-- -- -- -- > (10)} \]

Eqn (8)
\[ + 93.75 + \frac{1}{4} EI \theta_B + \frac{3}{2} EI \theta_C - \frac{3}{8} [EI \theta_B + EI \theta_C] = 0 \]
\[ + 93.75 - \frac{1}{8} EI \theta_B + \frac{9}{8} EI \theta_C = 0 \quad \text{-- -- -- -- > (11)} \]

Solving equations (10) & (11) we get \( EI \theta_B = 41.25 \)

By Equation (10)
\[ EI \theta_C = 8 \left[ -56.25 + \frac{9}{8} EI \theta_B \right] \]
\[ = 8 \left[ -56.25 + \frac{9}{8} 41.25 \right] = 78.75 \]
\[ \therefore EI \delta = EI \theta_B + EI \theta_C = 41.25 - 78.75 = -37.5 \]

Hence
\[ EI \theta_B = 41.25, \quad EI \theta_C = -78.75, \quad EI \delta = -37.5 \]
Substituting these values in slope deflection equations, we have

\[ M_{AB} = \frac{1}{2} (41.25) - \frac{3}{8} (-37.5) = +34.69 \text{ KNM} \]

\[ M_{BA} = 41.25 - \frac{3}{8} (-37.5) = +55.31 \text{ KNM} \]

\[ M_{BC} = -56.25 + \frac{1}{2} (41.25) + \frac{1}{4} (-78.75) = -55.31 \text{ KNM} \]

\[ M_{CB} = 93.75 + \frac{1}{2} (-78.75) + \frac{1}{4} (41.75) = +64.69 \text{ KNM} \]

\[ M_{CD} = -78.75 - \frac{3}{8} (-37.5) = -64.69 \text{ KNM} \]

\[ M_{DC} = \frac{1}{2} (-78.75) - \frac{3}{8} (-37.5) = -25.31 \text{ KNM} \]

**Reactions:** consider the free body diagram of beam and columns

Column AB:

\[ H_A = \frac{34.69 + 55.31}{4} = 22.5 \text{ KN} \]

Span BC:

\[ R_B = \frac{55.31 - 64.69 + 80 \times 3}{8} = 28.83 \text{ KN} \]

\[ \therefore R_C = 80 - R_B = 51.17 \]

Column CD:

\[ H_D = \frac{64.69 + 25.31}{4} = 22.5 \]

**Check:**

\[ \Sigma H = 0 \]

\[ H_A + H_D = 0 \]
Example: Frame ABCD is subjected to a horizontal force of 20 KN at joint C as shown in figure. Analyse and draw bending moment diagram.

Solution:

Frame is Symmetrical and unsymmetrical loaded hence there is a sway. Assume sway towards right

FEMS

\[ F_{AB} = F_{BA} = F_{BC} = F_{CB} = F_{CD} = F_{DC} = 0 \]

Slope deflection equations are
\[ M_{AB} = F_{AB} + \frac{2EI}{L} \left( 2\theta_A + \theta_B - \frac{3\delta}{L} \right) \]
\[ = \frac{2EI}{3} \left( \theta_B - \frac{3\delta}{3} \right) \]
\[ = \frac{2}{3} EI \theta_B - \frac{2}{3} EI \delta \]
\[ \quad \text{------------- > (1)} \]

\[ M_{BA} = F_{BA} + \frac{2EI}{L} \left( 2\theta_B + \theta_A - \frac{3\delta}{L} \right) \]
\[ = \frac{2EI}{3} \left( 2\theta_B - \frac{3\delta}{3} \right) \]
\[ = \frac{4}{3} EI \theta_B - \frac{2}{3} EI \delta \]
\[ \quad \text{------------- > (2)} \]

\[ M_{BC} = F_{BC} + \frac{2EI}{L} (2\theta_B + \theta_C) \]
\[ = \frac{2EI}{4} (2\theta_B + \theta_C) \]
\[ = EI \theta_B + 0.5 EI \theta_C \]
\[ \quad \text{------------- > (3)} \]

\[ M_{CB} = F_{CB} + \frac{2EI}{L} (2\theta_C + \theta_B) \]
\[ = \frac{2EI}{4} (2\theta_C + \theta_B) \]
\[ = EI \theta_C + 0.5 EI \theta_B \]
\[ \quad \text{------------- > (4)} \]

\[ M_{CD} = F_{CD} + \frac{2EI}{L} \left( 2\theta_C + \theta_B - \frac{3\delta}{L} \right) \]
\[ = \frac{2EI}{3} \left[ 2\theta_C - \frac{3\delta}{3} \right] \]
\[ = \frac{4}{3} EI \theta_C - \frac{2}{3} EI \delta \]
\[ \quad \text{------------- > (5)} \]

\[ M_{DC} = F_{DC} + \frac{2EI}{L} \left[ 2\theta_D + \theta_C - \frac{3\delta}{L} \right] \]
\[ = \frac{2EI}{3} \left[ \theta_C - \frac{3\delta}{3} \right] \]
\[ = \frac{2}{3} EI \theta_C - \frac{2}{3} EI \delta \]
\[ \quad \text{------------- > (6)} \]

The unknown are \( \theta_B, \theta_C \) & \( \delta \). Accordingly the boundary conditions are

1. \( M_{BA} + M_{BC} = 0 \)
2. \( M_{CB} + M_{CD} = 0 \)
3. \( H_A + H_D - 20 = 0 \)

i.e.,
\[ \frac{M_{AB} + M_{BA}}{3} + \frac{M_{CD} + M_{DC}}{3} - 20 = 0 \]
\[ M_{AB} + M_{BA} + M_{CD} + M_{DC} - 60 = 0 \]
Now $M_{BA} + M_{BC} = \frac{4}{3} E I \theta_B - \frac{2}{3} E I \delta + E I \theta_B + 0.5 E I \theta_C$

$$= \frac{7}{3} E I \theta_B + 0.5 E I \theta_C - \frac{2}{3} E I \delta = 0 \quad \cdots \quad (7)$$

and $M_{CB} + M_{CD} = E I \theta_C + 0.5 E I \theta_B + \frac{4}{3} E I \theta_C - \frac{2}{3} E I \delta$

$$= 0.5 E I \theta_B + \frac{7}{3} E I \theta_C - \frac{2}{3} E I \delta = 0 \quad \cdots \quad (8)$$

and $M_{AB} + M_{BA} + M_{CD} + M_{BC} - 60 = \frac{2}{3} E I \theta_B - \frac{2}{3} E I \delta + \frac{4}{3} E I \theta_B - \frac{2}{3} E I \delta + \frac{4}{3} E I \theta_C$

$$- \frac{2}{3} E I \delta + \frac{2}{3} E I \theta_C - \frac{2}{3} E I \delta - 60$$

$$= 2 E I \theta_B + 2 E I \theta_C - \frac{8}{3} E I \delta - 60 = 0 \quad \cdots \quad (9)$$

Solving (7), (8) & (9) we get

$$E I \theta_B = -8.18,$$

$$E I \theta_C = -8.18,$$

$$E I \delta = -34.77$$

Substituting the value of $\theta_B$, $\theta_C$ and $\delta$ in slope deflection equations

$$M_{AB} = \frac{2}{3} (-8.18) - \frac{2}{3} (-34.77) = +17.73 \text{ KNM}$$

$$M_{BA} = \frac{4}{3} (-8.18) - \frac{2}{3} (-34.77) = +12.27 \text{ KNM}$$

$$M_{BC} = 0 - 8.18 + 0.5 (-8.18) = -12.27 \text{ KNM}$$

$$M_{CB} = 0.5 (-8.18) - 8.18 = -12.27 \text{ KNM}$$

$$M_{CD} = \frac{4}{3} (-8.18) - \frac{2}{3} (-34.77) = +12.27 \text{ KNM}$$

$$M_{DC} = \frac{2}{3} (-8.18) - \frac{2}{3} (-34.77) = +17.73 \text{ KNM}$$
Reactions: Consider the free body diagram of the members

Member AB:
\[ H_A = \frac{17.73 + 12.27}{3} = 10 \text{ KN} \]

Member BC:
\[ R_c = \frac{12.27 + 12.27}{4} = 6.135 \text{ KN} \]
\[ \therefore R_b = -R_c = -6.135 \text{ KN} \quad \text{ -ve sign indicates direction of } R_b \text{ downward} \]

Member CD:
\[ H_d = \frac{-17.73 - 12.27}{3} = -10 \text{ KN} \quad \text{ -ve sign indicates the direction of } H_d \text{ is left to right} \]

Check: \[ \sum H = 0 \]
\[ H_A + H_D + P = 0 \]
\[ +10 + 10 - 20 = 0 \]
Hence okay
Example: Analyse the portal frame subjected to loads as shown. Also draw bending moment diagram.

The frame is symmetrical but loading is unsymmetrical. Hence there is a sway.
Assume sway towards right. In this problem $\theta_A = 0, \theta_B \neq 0, \theta_C = 0, \theta_D = 0$

FEMs:
$$F_{AB} = -\frac{w l}{12} = -\frac{10 \times 4^2}{12} = -13.33 \text{ KNM}$$
$$F_{BA} = +\frac{w l}{12} = +\frac{10 \times 4^2}{12} = +13.33 \text{ KNM}$$
$$F_{BC} = -\frac{w l}{8} = -\frac{90 \times 10}{8} = -112.5 \text{ KNM}$$
$$F_{CB} = \frac{w l}{8} = \frac{90 \times 10}{8} = +112.5 \text{ KNM}$$

Slope deflection equations:
$$M_{AB} = F_{AB} + \frac{2EI}{L} \left( 2\theta_A + \theta_B + \frac{3\delta}{L} \right)$$
$$= -13.33 + \frac{2EI}{4} \left( 0 + \theta_B - \frac{3\delta}{4} \right)$$
$$= -13.33 + 0.5 \theta_B - 0.375 \delta \delta$$  \hspace{1cm} \ldots (1)$$
$$M_{BA} = F_{BA} + \frac{2EI}{L} \left( 2\theta_B + \theta_A + \frac{3\delta}{L} \right)$$
$$= 13.33 + \frac{2EI}{4} \left( 2\theta_B + 0 - \frac{3\delta}{4} \right)$$

\[\text{BMD} \text{ in KNm}.\]
\[= 13.33 + \theta_B - 0.375 \theta \delta \]  

\[M_{bc} = F_{bc} + \frac{2EI}{L} (2\theta_B + \theta_C)\]

\[= -112.5 + \frac{2E3I}{10} (2\theta_B + \theta_C)\]

\[= -112.5 + 1.2 \theta_B + 0.6 \theta_C \]  

---------- > (3)

\[M_{cb} = F_{cb} + \frac{2EI}{L} (2\theta_C + \theta_B)\]

\[= +112.5 + \frac{2E3I}{10} (2\theta_C + \theta_B)\]

\[= 112.5 + 1.2 \theta_C + 0.6 \theta_B \]  

---------- > (4)

\[M_{cd} = F_{cd} + \frac{2EI}{L} (2\theta_C + \theta_B - 3\delta)\]

\[= 0 + \frac{2EI}{4} (2\theta_C + 0 - 3\delta)\]

\[= \theta_C - 0.375 \theta \delta \]  

---------- > (5)

\[M_{dc} = F_{dc} + \frac{2EI}{L} (2\theta_B + \theta_C - 3\delta)\]

\[= 0 + \frac{2EI}{4} (0 + 2\theta_C - 3\delta)\]

\[= 0.5 \theta_C - 0.375 \theta \delta \]  

---------- > (6)

There are 3 unknowns \( \theta_B \), \( \theta_C \), and \( \theta \delta \), accordingly the boundary conditions are

\[M_{ba} + M_{bc} = 0\]

\[M_{cb} + M_{dc} = 0\]

\[H_A + H_B + 40 = 0\]

Here \( H_A \times 4 = M_{ab} + M_{ba} - 10 \times 4 \times \frac{4}{2} \)

\[H_A = \frac{M_{ab} + M_{ba} - 80}{4}\]

and \( H_B \times 4 = M_{cd} + M_{bc} \)

\[H_B = \frac{M_{cd} + M_{bc}}{4}\]

\[\therefore \frac{M_{ab} + M_{ba} - 80}{4} + \frac{M_{cd} + M_{dc}}{4} + 40 = 0\]

\[M_{ab} + M_{ba} + M_{cd} + M_{dc} + 80 = 0\]

Now \( M_{ba} + M_{bc} = 0 \)
13.33EI\theta_B - 0.375EI\delta - 112.5 + 1.2EI\theta_B + 0.6EI\theta_C\theta_B = 0
2.2EI\theta_B + 0.6EI\theta_C - 0.375EI\delta - 99.17 = 0

and \ M_{OB} + M_{BC} = 0 \quad (4) + (5)

112.5 + 1.2EI\theta_C + 0.6EI\theta_B + EI\theta_C - 0.375EI\delta = 0
112.5 + 2.2EI\theta_C + 0.6EI\theta_B - 0.375EI\delta = 0

also \ M_{AB} + M_{BA} + M_{OB} + M_{BC} + 80 = 0

-13.33 + 0.5EI\theta_B - 0.375EI\delta + 13.33 + EI\theta_B - 0.375EI\delta + EI\theta_C - 0.375EI\delta

+ 0.5EI\theta_C - 0.375EI\delta + 80 = 0

1.5EI\theta_B + 1.5EI\theta_C - 1.5EI\delta + 80 = 0

By solving (7), (8) and (9) we get

El\theta_B = 72.65
El\theta_C = -59.64
El\delta = +66.34

Final moments:

\ M_{AB} = -13.33 + 0.5(72.65) - 0.375(66.34) = -1.88 \text{ KMN}
\ M_{BA} = +72.65 - 0.375(66.34) = 61.10 \text{ KMN}
\ M_{BC} = -112.5 + 1.2(72.65) + 0.6(-59.64) = -61.10 \text{ KMN}
\ M_{CB} = 112.5 + 1.2(-59.64) + 0.6(72.65) = 84.52 \text{ KMN}
\ M_{CD} = -59.64 - 0.375(66.34) = 84.52 \text{ KMN}
\ M_{DC} = 0.5(-59.64) - 0.375(66.34) = -54.70 \text{ KMN}

Reactions: Consider the free body diagrams of various members
Member AB:

\[ H_A = \frac{61.10 - 1.88 - 10 \times 4 \times 2}{4} \]

\[ = -5.195 \text{ KN} \]

-ve sign indicates direction of \( H_A \) is from right to left
Member BC:

\[ R_C = \frac{84.52 - 61.10 + 90 \times 5}{10} = 47.34 \text{ KN} \]

\[ \therefore R_B = 90 - R_C = 38.34 \text{ KN} \]

Member CD

\[ H_D = \frac{84.54 + 54.7}{4} = 34.81 \text{ KN} \]

Check

\[ \Sigma H = 0 \]

\[ H_A + H_D + 10 \times 4 = 0 \]

\[ -5.20 - 34.81 + 40 = 0 \]

Hence okay

Example: Analyse the portal frame and then draw the bending moment diagram
Solution:

Since the columns have different moment of inertia, it is an unsymmetrical frame.

Assume sway towards right

\[
F_{BC} = -\frac{WL}{8} = -\frac{80 \times 6}{8} = -60 \text{ KNM}
\]

\[
F_{CB} = +\frac{WL}{8} = +60 \text{ KNM}
\]

Here \( \theta_A = 0 \), \( \theta_B = 0 \)

Slope deflection equations

\[
M_{AB} = M_{AB} + \frac{2EI}{L} \left( 2\theta_A + \theta_B - \frac{3\delta}{L} \right)
\]

\[
= 0 + \frac{2EI}{4} \left( 0 + \theta_B - \frac{3\delta}{4} \right) = \frac{1}{2} EI \theta_B - \frac{3}{8} EI \delta \quad \text{----------> (1)}
\]

\[
M_{BA} = M_{BA} + \frac{2EI}{L} \left( 2\theta_B + \theta_A - \frac{3\delta}{L} \right)
\]

\[
= 0 + \frac{2EI}{4} \left( 2\theta_B + 0 - \frac{3\delta}{4} \right) = EI \theta_B - \frac{3}{8} EI \delta \quad \text{----------> (2)}
\]
\[ M_{bc} = F_{bc} + \frac{2EI}{L} \left( 2\theta_B + \theta_C \right) \]
\[ = -60 + \frac{2EI}{6} \left( 2\theta_B + \theta_C \right) = -60 + \frac{4}{3} EI\theta_B + \frac{2}{3} EI\theta_C \]  
\[ \quad \text{----------} > (3) \]

\[ M_{cb} = F_{cb} + \frac{2EI}{L} \left( 2\theta_C + \theta_B \right) \]
\[ = 60 + \frac{2EI}{6} \left( 2\theta_C + \theta_B \right) = 60 + \frac{2}{3} EI\theta_B + \frac{4}{3} EI\theta_C \]  
\[ \quad \text{----------} > (4) \]

\[ M_{cd} = F_{cd} + \frac{2EI}{L} \left( 2\theta_C + \theta_D - \frac{3\delta}{L} \right) \]
\[ = 0 + \frac{2EI}{4} \left( 2\theta_C + 0 - \frac{3\delta}{4} \right) = 2EI\theta_C - \frac{3}{4} EI\delta \]  
\[ \quad \text{----------} > (5) \]

\[ M_{dc} = F_{dc} + \frac{2EI}{L} \left( 2\theta_D + \theta_C - \frac{3\delta}{L} \right) \]
\[ = 0 + \frac{2EI}{4} \left( 0 + \theta_C - \frac{3\delta}{4} \right) = EI\theta_C - \frac{3}{4} EI\delta \]  
\[ \quad \text{----------} > (6) \]

In the above equation there are three unknowns \( \theta_B, \theta_C \) and \( \delta \), accordingly the boundary conditions are,

\[ M_{ba} + M_{bc} = 0 \]
\[ M_{cb} + M_{cd} = 0 \]  
\[ \quad \text{---} > \text{Joint conditions} \]
\[ H_A + H_B = 0 \]  
\[ \quad \text{---} > \text{Shear condition} \]

i.e, \( \frac{M_{ab} + M_{ba}}{4} + \frac{M_{cd} + M_{dc}}{4} = 0 \)
\[ \therefore \quad M_{ab} + M_{ba} + M_{cd} + M_{dc} = 0 \]

Now, \( M_{ba} + M_{bc} = EI\theta_B - \frac{3}{8} EI\delta - 60 + \frac{4}{3} EI\theta_B + \frac{2}{3} EI\theta_C = 0 \)
\[ = -60 + \frac{7}{3} EI\theta_B + \frac{2}{3} EI\theta_C - \frac{3}{8} EI\delta = 0 \]  
\[ \quad \text{----------} > (7) \]

And \( M_{cb} + M_{cd} = 60 + \frac{2}{3} EI\theta_B + \frac{4}{3} EI\theta_C + 2EI\theta_C - \frac{3}{4} EI\delta = 0 \)
\[ = \frac{2}{3} EI\theta_B + \frac{10}{3} EI\theta_C - \frac{3}{4} EI\delta + 60 = 0 \]  
\[ \quad \text{----------} > (8) \]

And \( M_{ab} + M_{bc} + M_{cd} + M_{dc} = \frac{1}{2} EI\theta_B - \frac{3}{8} EI\delta + EI\theta_B - \frac{3}{8} EI\delta + 2EI\theta_C - \frac{3}{4} EI\delta \)
\[ + EI\theta_C - \frac{3}{4} EI\delta \]
\[ = \frac{3}{2} EI\theta_B + 3EI\theta_C - \frac{9}{4} EI\delta = 0 \]  
\[ \quad \text{----------} > (9) \]
From (9) \[ EI\delta = \frac{4}{9} \left(\frac{3}{2} EI\theta_B + 3 EI\theta_C\right) \]

Substituting value of \( EI\delta \) in (7)

\[
\frac{7}{3} EI\theta_B + \frac{2}{3} EI\theta_C - \frac{3}{8} \left[\frac{4}{9} \left(\frac{3}{2} EI\theta_B + 3 EI\theta_C\right)\right] - 60 = 0
\]

\[
\frac{7}{3} EI\theta_B + \frac{2}{3} EI\theta_C - \frac{1}{4} EI\theta_B - \frac{1}{2} EI\theta_C - 60 = 0
\]

\[
\frac{25}{12} EI\theta_B + \frac{1}{6} EI\theta_C - 60 = 0
\]

\[ \text{--------} \rightarrow (10) \]

Substituting value of \( EI\delta \) in (8)

\[
\frac{2}{3} EI\theta_B + \frac{10}{3} EI\theta_C - \frac{3}{8} \left[\frac{4}{9} \left(\frac{3}{2} EI\theta_B + 3 EI\theta_C\right)\right] + 60 = 0
\]

\[
\frac{2}{3} EI\theta_B + \frac{10}{3} EI\theta_C - \frac{1}{2} EI\theta_B - EI\theta_C + 60 = 0
\]

\[
\frac{1}{6} EI\theta_B + \frac{7}{3} EI\theta_C + 60 = 0
\]

\[ \text{--------} \rightarrow (11) \]

Solving (10) & (11) we get \( EI\theta_B = 31.03 \)

By Equation (11)

\[
EI\theta_B = \frac{3}{7} \left[\frac{1}{6} EI\theta_C + 60\right]
\]

Now

\[ EI\delta = \frac{4}{9} \left(\frac{3}{2} EI\theta_B + 3 EI\theta_C\right) = -16.55 \]

Now

\( EI\theta_B = 31.03, \ EI\theta_C = -27.3, \ EI\delta = -16.55 \)

Substituting these values in slope deflection equations,

The final moments are:

\[
M_{AB} = \frac{1}{2} (31.03) - \frac{3}{8} (-16.55) = 21.72 \text{ KNM}
\]

\[
M_{BA} = 31.03 - \frac{3}{8} (-16.55) = 37.24 \text{ KNM}
\]

\[
M_{BC} = -60 + \frac{4}{3} (31.03) + \frac{2}{3} (-27.93) = -37.25 \text{ KNM}
\]

\[
M_{CB} = +60 + \frac{2}{3} (31.03) + \frac{4}{3} (-27.93) = +43.43 \text{ KNM}
\]

\[
M_{CD} = 2 (-27.93) - \frac{3}{4} (-16.55) = -43.45 \text{ KNM}
\]

\[
M_{DC} = -27.93 - \frac{3}{4} (-16.55) = -15.52 \text{ KNM}
\]
Reactions: consider the free body diagram of beam and columns

Column AB:
\[ H_A = \frac{37.25 + 21.72}{4} = 14.74 \text{ KN} \]

Beam BC:
\[ R_B = \frac{37.25 - 43.45 + 80 \times 3}{6} = 38.97 \text{ KN} \]
\[ R_C = 80 - R_B = 41.03 \]

Column CD:
\[ H_D = \frac{43.45 + 15.52}{4} = 14.74 \text{ KN} \]

Check:
\[ \Sigma H = 0 \]
\[ H_A + H_D = 0 \]
\[ 14.74 - 14.74 = 0 \]

Hence okay
**Ex:** Portal frame shown is fixed at ends A and D, the joint B is rigid and joint C is hinged. Analyse the frame and draw BMD.

**Solution:**

FEM's:

\[
\begin{align*}
F_{BC} &= -\frac{WL}{8} = -\frac{80 \times 6}{8} = -60 \text{ KNM} \\
F_{CB} &= +\frac{WL}{8} = \frac{80 \times 6}{8} = +60 \text{ KNM}
\end{align*}
\]

Here \( \theta_A = 0, \ \theta_D = 0, \ \theta_B \neq 0, \ \theta_{CB} \neq 0, \ \theta_{CD} \neq 0 \)

Since C is hinged member CB and CD will rotate independently. Also the frame is unsymmetrical, will also have sway. Let the sway be towards right.

The slope deflections are:

\[
M_{AB} = F_{AB} + \frac{2EI}{L} \left( 2\theta_A + \theta_B - \frac{3\delta}{L} \right)
\]

\[
= 0 + \frac{2EI}{4} \left( 0 + \theta_B - \frac{3\delta}{4} \right)
\]

\[
= \frac{1}{2} EI \theta_B - \frac{3}{8} EI \delta \quad \text{----> (1)}
\]
\[ M_{BA} = F_{BA} + \frac{2EI}{L} \left( 2\theta_B + \theta_A - \frac{3\delta}{L} \right) \]
\[ = 0 + \frac{2EI}{4} \left( 2\theta_B + 0 - \frac{3\delta}{4} \right) \]
\[ = EI\theta_B - \frac{3}{8}EI\delta \quad \text{----> (2)} \]

\[ M_{BC} = F_{BC} + \frac{2EI}{L} \left( 2\theta_B + \theta_C - \frac{3\delta}{L} \right) \]
\[ = -60 + \frac{2EI}{6} (2\theta_B + \theta_{CB}) \]
\[ = -60 + \frac{4}{3}EI\theta_B + \frac{2}{3}EI\theta_{CB} \quad \text{----> (3)} \]

\[ M_{CB} = F_{CB} + \frac{2EI}{L} (2\theta_{CB} + \theta_B) \]
\[ = +60 + \frac{2EI}{6} (2\theta_{CB} + \theta_B) \]
\[ = +60 + \frac{4}{3}EI\theta_{CB} + \frac{2}{3}EI\theta_B \quad \text{----> (4)} \]

\[ M_{CD} = F_{CD} + \frac{2EI}{L} \left( 2\theta_{CD} + \theta_B - \frac{3\delta}{L} \right) \]
\[ = 0 + \frac{2EI}{4} \left( 2\theta_{CD} + 0 - \frac{3\delta}{4} \right) \]
\[ = EI\theta_{CD} - \frac{3}{8}EI\delta \quad \text{----> (5)} \]

\[ M_{DC} = F_{DC} + \frac{2EI}{L} \left( 2\theta_B + \theta_{CD} - \frac{3\delta}{L} \right) \]
\[ = 0 + \frac{2EI}{4} \left( 0 + \theta_{CD} - \frac{3\delta}{4} \right) \]
\[ = \frac{1}{2}EI\theta_{CD} - \frac{3}{8}EI\delta \quad \text{----> (6)} \]

In the above equations \( \theta_B, \theta_{CB}, \theta_{CD} \) and \( \delta \) are unknowns. According the boundary conditions are

I. \( M_{BA} + M_{BC} = 0 \),

II. \( M_{CB} = 0 \),

III. \( M_{CD} = 0 \),

IV. \( H_A + H_D = 0 \)

i.e., \[ \frac{M_{AB} + M_{BA}}{4} + \frac{M_{CD} + M_{DC}}{4} = 0 \]

\[ \therefore M_{AB} + M_{BA} = M_{CD} + M_{DC} = 0 \]

Now using the boundary conditions:
\[ M_{BA} + M_{BC} = \Theta \theta_B - \frac{3}{8} \Theta \delta - 60 + \frac{4}{3} \Theta \theta_B + \frac{2}{3} \Theta \theta_{CB} \]
\[ = \frac{7}{3} \Theta \theta_B + \frac{2}{3} \Theta \theta_{CB} - \frac{3}{8} \Theta \delta - 60 = 0 \quad \Rightarrow (7) \]

\[ M_{CB} = \frac{2}{3} \Theta \theta_B + \frac{4}{3} \Theta \theta_{CB} + 60 = 0 \quad \Rightarrow (8) \]

\[ M_{CD} = \Theta \theta_{CD} - \frac{3}{8} \Theta \delta = 0 \quad \Rightarrow (9) \]

\[ M_{AB} + M_{BA} + M_{CD} + M_{DC} = \frac{1}{2} \Theta \theta_B - \frac{3}{8} \Theta \delta + \Theta \theta_B - \frac{3}{8} \Theta \delta + \Theta \theta_{CD} - \frac{3}{8} \Theta \delta + \frac{1}{2} \Theta \theta_{CD} - \frac{3}{8} \Theta \delta = 0 \]
\[ = \frac{3}{2} \Theta \theta_B + \frac{3}{2} \Theta \theta_{CD} - \frac{3}{2} \Theta \delta = 0 \quad \Rightarrow (10) \]

From (9) \[ \Theta \theta_{CD} = \frac{3}{8} \Theta \delta \quad \Rightarrow (11) \]

Sub in (10)
\[ \therefore M_{AB} + M_{BA} + M_{CD} + M_{DC} = \frac{3}{2} \Theta \theta_B + \frac{3}{2} \left( \frac{3}{8} \Theta \delta \right) - \frac{3}{2} \Theta \delta = 0 \]
\[ = \frac{3}{2} \Theta \theta_B - \frac{15}{16} \Theta \delta = 0 \quad \Rightarrow (12) \]

Equation (12) gives \[ \Theta \delta = \frac{16}{15} \times \frac{3}{2} \Theta \theta_B = \frac{8}{5} \Theta \theta_B \quad \Rightarrow (13) \]

Substituting in Equation (7)
\[ M_{BA} + M_{BC} = \frac{7}{3} \Theta \theta_B + \frac{2}{3} \Theta \theta_{CB} - \frac{3}{8} \left( \frac{8}{5} \Theta \theta_B \right) - 60 = 0 \]
\[ = \left( \frac{7}{3} - \frac{3}{5} \right) \Theta \theta_B + \frac{2}{3} \Theta \theta_{CB} - 60 \]
\[ = \frac{26}{15} \Theta \theta_B + \frac{2}{3} \Theta \theta_{CB} - 60 = 0 \quad \Rightarrow (14) \]

Substituting in Equation (8) and multiplying equation (14) by 2 we have
\[ \frac{2}{3} \Theta \theta_B + \frac{4}{3} \Theta \theta_{CB} + 60 = 0 \]
\[ \frac{52}{15} \Theta \theta_B + \frac{4}{3} \Theta \theta_{CB} - 120 = 0 \]

\[ \frac{42}{15} \Theta \theta_B + 180 = 0 \]
\[ \therefore \Theta \theta_B = 180 \times \frac{15}{42} = 64.29 \]
From (13) \( \theta = \frac{8}{5} \delta \theta_B = \frac{8}{5} \times 64.29 = 102.864 \)

From (11) \( \theta_{CD} = \frac{3}{8} \delta \)

\( \delta = 38.574 \)

From (7) \( \theta_{CB} = -\frac{3}{2} \left( \frac{7}{3} \delta \theta_B - \frac{3}{8} \delta \right) \)

\( = -\frac{3}{2} \left( \frac{7}{3} \times 64.29 - \frac{3}{8} \times 102.864 \right) \)

\( = -77.165 \)

\( \delta \theta_B = 64.29, \; \theta_{CB} = -77.165, \; \delta \theta_{CD} = 38.57, \; \delta \delta = 102.864 \)

:. Final Moments are

\( M_{AB} = \frac{1}{2} (64.29) - \frac{3}{8} (102.864) = -6.42 \text{ KNM} \)

\( M_{BA} = 64.29 - \frac{3}{8} (102.864) = 25.72 \text{ KNM} \)

\( M_{BC} = -60 + \frac{4}{3} (64.29) + \frac{2}{3} (-77.165) = -25.72 \text{ KNM} \)

\( M_{CB} = +60 + \frac{4}{3} (-77.165) + \frac{2}{3} (64.29) = 0 \)

\( M_{CD} = 38.574 - \frac{3}{8} (102.864) = 0 \)

\( M_{DC} = \frac{1}{2} (38.574) - \frac{3}{8} (102.864) = -19.29 \text{ KNM} \)

**Reactions:** Consider the free body diagram of various members
Column AB:

\[ H_A = \frac{25.72 - 6.42}{4} = 4.825 \text{ KN} \]

Beam BC:

\[ R_B = \frac{25.72 + (80 \times 3)}{6} = 44.29 \text{ KN} \]
\[ R_C = 80 - 44.29 = 35.71 \text{ KN} \]

Column CD:

\[ H_D = \frac{19.28}{4} = 4.82 \text{ KN} \]

Check:

\[ \Sigma H = 0 \]
\[ H_A + H_D = 0 \]

Hence okay.
Example: Analyse the portal frame shown in figure the deflection method and then draw the bending moment diagram

Solution:

The frame is unsymmetrical, hence there is a sway. Let the sway be towards right.

\[ \theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D = 0 \]

FEMS:

\[ F_{BC} = -\frac{20 \times 5^2}{12} = -41.67 \text{ KNM} \]
\[ F_{CB} = +\frac{20 \times 5^2}{12} = +41.67 \text{ KNM} \]
\[ F_{CE} = -15 \times 2 = -30 \text{ KNM} \]

Slope deflection equations
In the above equation there are three unknowns $\theta_B$, $\theta_C$ and $\delta$, accordingly the boundary conditions are,

$$\begin{align*}
M_{BA} + M_{BC} &= 0 \\
M_{CB} + M_{CD} + M_{CE} &= 0 \\
H_A + H_B &= 0
\end{align*}$$

i.e, $M_{AB} + M_{BA} + M_{CD} + M_{DC} = 0$

Now,

$$\begin{align*}
M_{BA} + M_{BC} &= 0 \\
\Delta \theta_B - 0.375EI\delta - 141.67 + 1.2EI \theta_B + 0.6EI \theta_C &= 0 \\
= 2.2EI \theta_B + 0.6EI \theta_C - 0.375EI\delta - 41.67 &= 0
\end{align*}$$  \quad \text{--------- (7)}$$

And $M_{CB} + M_{CD} = 41.67 + 1.2EI \theta_C + 0.6EI \theta_B + EI \theta_c - 0.375EI\delta - 30 = 0$

$$\begin{align*}
= 0.6EI \theta_b + 2.2EI \theta_c - 0.375EI\delta + 11.67 &= 0
\end{align*}$$  \quad \text{--------- (8)}$$
\[ \therefore M_{AB} + M_{BC} + M_{CD} + M_{DC} = 0 \]

\[ 0.5EI\theta_B - 0.375EI\delta + EI\theta_B - 0.375EI\delta + 2EI\theta_C - 0.375EI\delta + 0.5EI\theta_C - 0.375EI\delta = 0 \]

\[ 1.5EI\theta_B + 1.5EI\theta_C - 1.5EI\delta = 0 \]

Solving the above equations we get, \( EI\theta_B = 23.98 \), \( EI\theta_C = -9.36 \), \( EI\delta = 14.62 \)

Substituting these values in slope deflection equations, we have
\[ M_{AB} = 0.5(23.98) - 0.375(14.62) = 6.50 \text{ KNM} \]
\[ M_{BA} = 23.98 - 0.375(14.62) = +18.50 \text{ KNM} \]
\[ M_{BC} = -41.67 + 1.2(23.98) + 0.6(-9.36) = -18.51 \text{ KNM} \]
\[ M_{CB} = +41.67 + 1.2(-9.36) + 0.6(23.98) = +44.83 \text{ KNM} \]
\[ M_{CD} = -9.36 - 0.375(+14.62) = -14.84 \text{ KNM} \]
\[ M_{DC} = 0.5 \times -9.36 - 0.375 \times 14.62 = -10.16 \text{ KNM} \]
\[ M_{CE} = -30 \text{ KNM} \]

**Reactions:** consider the free body diagram of beam and columns
Column AB:

\[ H_A = \frac{18.5 + 6.5}{4} = 6.25 \text{ KN} \]

Span BC:

\[ R_C = \frac{44.83 - 18.5 + 20 \times 5 \times 2.5}{5} = 55.27 \text{ KN} \]

\[ R_B = 20 \times 5 - R_C = 44.73 \]

Column CD:

\[ H_D = \frac{10.16 + 14.84}{4} = 6.25 \]

Check:

\[ \Sigma H = 0 \]

\[ H_A + H_D = 0 \]

\[ \Sigma = 0 \]

Hence okay

Example: Analyse the portal frame shown and then draw bending moment diagram.
Solution:

It is an unsymmetrical problem hence there is a sway be towards right
\( \theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D = 0 \)

FEMs:

\[
F_{BC} = - \frac{w l^2}{12} = - \frac{20 \times 5^2}{12} = - 41.67 \text{ KNM}
\]

\[
F_{CB} = \frac{w l^2}{12} = \frac{20 \times 5^2}{12} = + 41.67 \text{ KNM}
\]

Slope deflection equations:

\[
M_{AB} = F_{AB} + \frac{2EI}{L} \left( 2\theta_A - \frac{3\delta}{L} \right)
\]

\[
= 0 + \frac{2EI}{3} \left( 0 + \frac{3\delta}{3} \right)
\]

\[
= \frac{2}{3} EI\theta_B - \frac{2}{3} EI\delta
\]

\[
M_{BA} = F_{BA} + \frac{2EI}{L} \left( 2\theta_B - \frac{3\delta}{L} \right)
\]

\[
= 0 + \frac{2EI}{3} \left( 2\theta_B + 0 - \frac{3\delta}{3} \right)
\]

\[
= \frac{4}{3} EI\theta_B - \frac{2}{3} EI\delta
\]

\[
M_{BC} = F_{BC} + \frac{2EI}{L} \left( 2\theta_B + \theta_C - \frac{3\delta}{L} \right)
\]

\[
= - 41.67 + \frac{2EI \times 1.5I}{5} \left( 2\theta_B + \theta_C \right)
\]

\[
= - 41.67 + \frac{6}{5} EI\theta_B + \frac{3}{5} EI\theta_C
\]

\[
M_{CB} = F_{CB} + \frac{2EI}{L} \left( 2\theta_C + \theta_B - \frac{3\delta}{L} \right)
\]

\[
= 41.67 + \frac{2EI \times 1.5I}{5} \left( 2\theta_C + \theta_B - 0 \right)
\]

\[
= 41.67 + 1.2 EI\theta_C + 0.6 EI\theta_B
\]
\[
M_{cd} = F_{cd} + \frac{2EI}{L} \left( 2\theta_c + \theta_d - \frac{3\delta}{L} \right) \\
= 0 + \frac{2EI}{4} \left( 2\theta_c + 2\theta_c + 0 - \frac{3\delta}{4} \right) \\
= EI \theta_c - 0.375 EI \delta \\
\text{-------------} \quad (5)
\]

\[
M_{dc} = F_{dc} + \frac{2EI}{L} \left( 2\theta_d + \theta_c - \frac{3\delta}{L} \right) \\
= 0 + \frac{2EI}{4} \left( 0 + \theta_c - \frac{3\delta}{4} \right) \\
= 0.5 EI \theta_c - 0.375 EI \delta \\
\text{-------------} \quad (6)
\]

In the above equations there are three unknowns \( \theta_b, \theta_c \) and \( \delta \) and accordingly the Boundary conditions are:

\[
M_{ba} + M_{bc} = 0 \\
M_{cb} + M_{cd} = 0 \\
H_A + H_B = 0
\]

i.e \( \frac{M_{ab} + M_{ba}}{3} + \frac{M_{cd} + M_{dc}}{4} = 0 \)

\[
\therefore \quad 4(M_{ab} + M_{ba}) + 3(M_{cd} + M_{dc}) = 0
\]
Now

\[ M_{BA} + M_{BC} = 0 \]
\[ \frac{4}{3} EI \theta_B - \frac{2}{3} EI \delta + \frac{6}{5} EI \theta_B + \frac{3}{5} EI \theta_C - 41.67 \]
\[ 2.53EI \theta_B + \frac{3}{5} EI \theta_C - \frac{2}{3} EI \delta - 41.67 = 0 \text{ } \cdots \text{ } (7) \]

\[ M_{CB} + M_{CD} = 0 \]
\[ 41.67 + 1.2EI \theta_C + 0.6EI \theta_B + EI \theta_C - 0.375EI \delta = 0 \]
\[ 41.67 + 2.2EI \theta_C + 0.6EI \theta_B - 0.375EI \delta = 0 \text{ } \cdots \text{ } (8) \]

\[ \frac{M_{AB} + M_{BA}}{3} + \frac{M_{CD} + M_{DC}}{4} = 0 \]
\[ 4 \left( \frac{2}{3} EI \theta_B - \frac{2}{3} EI \delta + \frac{4}{3} EI \theta_B - \frac{2}{3} EI \delta \right) + 
3 \left[ EI \theta_C - 0.375EI \delta + 0.5EI \theta_C - 0.375EI \delta \right] = 0 \]
\[ \frac{8}{3} EI \theta_B - \frac{8}{3} EI \delta + \frac{16}{3} EI \theta_B - \frac{8}{3} EI \delta + 4.5EI \theta_C - 2.25EI \delta = 0 \]
\[ 8EI \theta_B + 4.5EI \theta_C - 7.53EI \delta = 0 \text{ } \cdots \text{ } (9) \]

By solving (7), (8) and (9) we get

\[ EI \theta_B = +25.46 \]
\[ EI \theta_C = -23.17 \]
\[ EI \delta = +12.8 \]

Final moments:

\[ M_{AB} = \frac{2}{3} \times 25.46 - \frac{2}{3} \times 12.8 = 8.44 \text{ } \text{KNM} \]
\[ M_{BA} = \frac{4}{3} \times 25.46 - \frac{2}{3} \times 12.8 = 25.40 \text{ } \text{KNM} \]
\[ M_{BC} = \frac{6}{5} \times 25.46 + \frac{3}{5} \times (-23.17) - 41.67 = -25.40 \text{ } \text{KNM} \]
\[ M_{CB} = 41.67 + 1.2 \times (-23.17) + 0.60 \times 20.46 = 28.50 \text{ } \text{KNM} \]
\[ M_{CD} = -23.70 - 0.375 \times 12.80 = -28.50 \text{ } \text{KNM} \]
\[ M_{DC} = 0.5 \times -23.70 - 0.375 \times 12.80 = -16.65 \text{ } \text{KNM} \]
Reactions: Consider the free body diagram

Member AB:

\[ H_A = \frac{25.40 + 8.44}{3} = 11.28 \text{ KN} \]

Member BC:

\[ R_C = \frac{28.5 - 20.30 + 20 \times 5 \times \frac{5}{2}}{2} = 51.64 \text{ KN} \]

\[ :. R_B = 20 \times 5 - 51.64 = 48.36 \text{ KN} \]

Member CD:

\[ H_D = \frac{28.5 + 16.65}{4} = 11.28 \text{ KN} \]

Check:

\[ \Sigma H = 0 \]

\[ H_A + H_D = 0 \]

Satisfied, hence okay
**Example:** A portal frame having different column heights are subjected for forces as shown in figure. Analyse the frame and draw bending moment diagram.

**Solution:-**

It is an unsymmetrical problem
\( \theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D = 0 \), hence there is a sway be towards right.

FEMs:

\[
F_{AB} = -\frac{WI}{8} = -\frac{30 \times 4}{8} = -15 \text{ KNM}
\]

\[
F_{BA} = +\frac{WI}{8} = +\frac{30 \times 4}{8} = +15 \text{ KNM}
\]

\[
F_{BC} = -\frac{WI}{8} = -\frac{60 \times 4}{8} = -30 \text{ KNM}
\]

\[
F_{CB} = +\frac{WI}{8} = +\frac{60 \times 4}{8} = +30 \text{ KNM}
\]

\[
F_{CD} = F_{DC} = 0
\]

Slope deflection equations:
\[ M_{AB} = F_{AB} + \frac{2EI}{L} \left( 2\theta_A + \theta_B - \frac{3\delta}{L} \right) \]
\[ = -15 + \frac{2E2I}{4} \left( 0 + \theta_B - \frac{3\delta}{4} \right) \]
\[ = -15 + 2E\theta_B - 0.75E\delta \] \hspace{1cm} \cdots \hspace{1cm} (1) \\

\[ M_{BA} = F_{BA} + \frac{2EI}{L} \left( 2\theta_B + \theta_A - \frac{3\delta}{L} \right) \]
\[ = 15 + \frac{2E2I}{4} \left( 2\theta_B + 0 - \frac{3\delta}{4} \right) \]
\[ = 15 + 2E\theta_B - 0.75E\delta \] \hspace{1cm} \cdots \hspace{1cm} (2) \\

\[ M_{BC} = F_{BC} + \frac{2EI}{L} \left( 2\theta_B + \theta_C \right) \]
\[ = -30 + \frac{2E2I}{4} \left( 2\theta_B + \theta_C \right) \]
\[ = -30 + 2E\theta_B + E\theta_C \] \hspace{1cm} \cdots \hspace{1cm} (3) \\

\[ M_{CB} = F_{CB} + \frac{2EI}{L} \left( 2\theta_C + \theta_B \right) \]
\[ = 30 + \frac{2E2I}{4} \left( 2\theta_C + \theta_B \right) \]
\[ = 30 + 2E\theta_C + E\theta_B \] \hspace{1cm} \cdots \hspace{1cm} (4) \\

\[ M_{CD} = F_{CD} + \frac{2EI}{L} \left( 2\theta_C + \theta_D - \frac{3\delta}{L} \right) \]
\[ = 0 + \frac{2EI}{3} \left( 2\theta_C + 0 - \frac{3\delta}{3} \right) \]
\[ = \frac{4}{3}E\theta_C - \frac{2}{3}E\delta \] \hspace{1cm} \cdots \hspace{1cm} (5) \\

\[ M_{DC} = F_{DC} + \frac{2EI}{L} \left( 2\theta_D + \theta_C - \frac{3\delta}{L} \right) \]
\[ = 0 + \frac{2EI}{3} \left( 0 + \theta_C - \frac{3\delta}{3} \right) \]
\[ = \frac{2}{3}E\theta_C - \frac{2}{3}E\delta \] \hspace{1cm} \cdots \hspace{1cm} (6) \\

There are three unknowns, EI, \( \theta_B \), \( \theta_C \) & \( \theta_C \), accordingly the Boundary conditions are

\[ M_{BA} + M_{BC} = 0 \]
\[ M_{CB} + M_{CD} = 0 \]
\[ H_A + H_B + 30 = 0 \]
\[ \text{i.e., } \frac{M_{AB} + M_{BA}}{4} - \frac{M_{CD} + M_{DC}}{3} + 30 = 0 \]
\[ 3(M_{AB} + M_{BA}) + 4(M_{CD} + M_{DC}) + 180 = 0 \]

Now

\[ M_{BA} + M_{BC} = 15 + 2E\theta_B - 0.75E\delta - 30 + 2E\theta_B + E\theta_C \]
\[ = 4E\theta_B + E\theta_C - 0.75E\delta - 15 = 0 \] \hspace{1cm} \cdots \hspace{1cm} (7)
\[ M_{CB} + M_{CD} = +30 + 2EI\theta_c + EI\theta_b + \frac{4}{3} EI\theta_c - \frac{2}{3} EI\delta \]
\[ = EI\theta_b + \frac{10}{3} EI\theta_c - \frac{2}{3} EI\delta + 30 = 0 \quad \cdots \quad (8) \]

\[ 3(M_{AB} + M_{BA}) + 4(M_{CD} + M_{DC}) + 180 = 3 \left( -15 + EI\theta_b - 0.75EI\delta + 15 + 2EI\theta_b - 0.75EI\delta \right) \]
\[ + 4 \left( \frac{4}{3} EI\theta_c - \frac{2}{3} EI\delta + \frac{2}{3} EI\theta_c - \frac{2}{3} EI\delta \right) + 180 \]
\[ = 9EI\theta_b + 8 EI\theta_c - 9.833EI\delta + 180 = 0 \quad \cdots \quad (9) \]

By solving (7), (8) & (9) we get

\[ EI\theta_b = +9.577 \]
\[ EI\theta_c = -7.714 \]
\[ EI\delta = +20.795 \]

Substituting these values in the slope deflection equations we get

\[ M_{AB} = -15 + 9.577 - 0.75(20.795) = -21.01 \text{ KNM} \]
\[ M_{BA} = +15 + 2 (9.577) - 0.75 (20.795) = 18.55 \text{ KNM} \]
\[ M_{BC} = -30 + 2 (9.577) - 7.714 = -18.55 \text{ KNM} \]
\[ M_{CB} = 30 + 2 (-7.714) + 9.577 = 24.15 \text{ KNM} \]
\[ M_{CD} = \frac{4}{3} (-7.714) - \frac{2}{3} (20.795) = -24.15 \text{ KNM} \]
\[ M_{DC} = \frac{2}{3} (-7.714) - \frac{2}{3} (20.795) = -19.00 \text{ KNM} \]

**Reactions:** Consider free body diagrams of the members
Member AB:

\[ H_A = \frac{18.55 - 21.01 - 30 \times 2}{4} = -15.615 \text{ KN} \]

-ve sign indicates the direction of \( H_A \) is from right to left.

Member BC:

\[ R_B = \frac{18.55 + 60 \times 2 - 24.15}{4} = 28.60 \text{ KN} \]
\[ R_C = 60 - R_B = 28.60 = 31.40 \text{ KN} \]

Member CD:

\[ H_D = \frac{19 + 24.15}{3} = 14.38 \text{ KN} \]

Check:

\[ \Sigma H = 0 \]
\[ H_A + H_D + 30 = 0 \]
\[ -15.62 - 14.38 + 30 = 0 \]

Hence okay

**Example:** Analyse the frame using slope deflection method and draw the Bending Moment Diagram.
Solution: Assume sway towards right

It can be observed from figure in that direction of moments due to sway in member AB are anticlockwise and that for member CD are clockwise. Wise shall be taken to incorporate the same in the slope deflection equation.
Slope deflection equations are:

\[ M_{AB} = F_{AB} + \frac{2EI}{L} \left( 2\theta_A + \theta_B - \frac{3\delta}{L} \right) \]
\[ = \frac{2EI}{3} \left( \theta_B - \frac{3\delta}{3} \right) \]
\[ = \frac{2}{3} EI \theta_B - \frac{2}{3} EI \delta \quad \text{------- > (1)} \]

\[ M_{BA} = F_{BA} + \frac{2EI}{L} \left( 2\theta_B + \theta_A - \frac{3\delta}{L} \right) \]
\[ = \frac{2EI}{3} \left( 2\theta_B - \frac{3\delta}{3} \right) \]
\[ = \frac{4}{3} EI \theta_B - \frac{2}{3} EI \delta \quad \text{------- > (2)} \]

\[ M_{BC} = F_{BC} + \frac{2EI}{L} \left( 2\theta_B + \theta_C \right) \]
\[ = -32 + \frac{2EI}{4} \left( 2\theta_B + \theta_C \right) \]
\[ = -32 + 2EI \theta_B + EI \theta_C \quad \text{------- > (3)} \]

\[ M_{CB} = F_{CB} + \frac{2EI}{L} \left( 2\theta_C + \theta_B \right) \]
\[ = +32 + \frac{2EI}{4} \left( 2\theta_C + \theta_B \right) \]
\[ = +32 + 2EI \theta_C + EI \theta_B \quad \text{------- > (4)} \]

\[ M_{CD} = F_{CD} + \frac{2EI}{L} \left( 2\theta_C + \theta_D + \frac{3\delta}{L} \right) \]
\[ = \frac{2EI}{3} \left[ 2\theta_C + \frac{3\delta}{3} \right] \]
\[ = \frac{4}{3} EI \theta_C + \frac{2}{3} EI \delta \quad \text{------- > (5)} \]
\[ M_{\text{DC}} = F_{\text{DC}} + \frac{2EI}{L} \left[ 2\theta_c + \theta_b + \frac{3\delta}{L} \right] \]
\[ = \frac{2EI}{3} \left( \theta_c + \frac{3\delta}{3} \right) \]
\[ = \frac{2}{3} EI \theta_c + \frac{2}{3} EI \delta \]

---------- > (6)

The unknown are \( \theta_b, \theta_c, \delta \). Accordingly, the boundary conditions are

\[ M_{\text{BA}} + M_{\text{BC}} = 0 \]
\[ M_{\text{CB}} + M_{\text{CD}} = 0 \]
\[ H_A - H_D + 30 = 0 \]

i.e.
\[ \frac{M_{\text{AB}} + M_{\text{BA}}}{3} - \frac{M_{\text{CD}} + M_{\text{DC}}}{3} + 30 = 0 \]
\[ M_{\text{AB}} + M_{\text{BA}} - M_{\text{CD}} - M_{\text{DC}} + 90 = 0 \]

Now,
\[ M_{\text{BA}} + M_{\text{BC}} = \frac{4}{3} EI \theta_b - \frac{2}{3} EI \delta - 32 + 2EI \theta_b + EI \theta_c \]
\[ = \frac{10}{3} EI \theta_b + EI \theta_c - \frac{2}{3} EI \delta - 32 = 0 \]  ---------- > (7)

\[ M_{\text{CB}} + M_{\text{CD}} = +32 + 2EI \theta_c + EI \theta_b + \frac{4}{3} EI \theta_c + \frac{2}{3} EI \delta \]
\[ = EI \theta_b + \frac{10}{3} EI \theta_c + \frac{2}{3} EI \delta + 32 = 0 \]  ---------- > (8)

\[ M_{\text{AB}} + M_{\text{BA}} - M_{\text{CD}} - M_{\text{DC}} + 90 = \frac{2}{3} EI \theta_b - \frac{2}{3} EI \delta + \frac{4}{3} EI \theta_b - \frac{2}{3} EI \sigma - \frac{4}{3} EI \theta_c \]
\[ - \frac{2}{3} EI \delta - \frac{2}{3} EI \theta_c - \frac{2}{3} EI \delta + 90 \]
\[ = 2EI \theta_b - 2EI \theta_c - \frac{8}{3} EI \delta + 90 \]
\[ = EI \theta_b - EI \theta_c - \frac{4}{3} EI \delta + 45 = 0 \]  ---------- > (9)

From (7) & (9)
\[ \frac{20}{3} EI \theta_b + 2EI \theta_c - \frac{4}{3} EI \delta - 64 = 0 \]
\[ EI \theta_b - EI \theta_c - \frac{4}{3} EI \delta + 45 = 0 \]
\[ \frac{17}{3} EI \theta_b + 3EI \theta_c - 109 = 0 \]  ---------- > (10)

By (8) and (9)
\[ 2EI \theta_B + \frac{20}{3} EI \theta_C + \frac{4}{3} EI \delta + 64 = 0 \]
\[ EI \theta_B - EI \theta_C - \frac{4}{3} EI \gamma + 45 = 0 \]
\[ 3EI \theta_B + \frac{17}{3} EI \theta_C + 109 = 0 \]

By (10) & (11)
\[ \frac{17}{3} EI \theta_B + 3EI \theta_C - 109 = 0 \]
\[ \frac{27}{17} EI \theta_B + 3EI \theta_C + 57.71 = 0 \]
\[ \frac{208}{17 \times 3} EI \theta_B - 166.71 = 0 \]
\[ EI \theta_B = \frac{166.71 \times 17 \times 3}{208} = +40.88 \]

From (10)
\[ EI \theta_C = \frac{1}{3} \left[ 109 - \frac{17}{3} EI \theta_B \right] = -40.88 \]

From (9)
\[ EI \delta = \frac{3}{4} \left[ EI \theta_B - EI \theta_C + 45 \right] \]
\[ = \frac{3}{4} \left[ 40.88 - (-40.88) + 45 \right] = +95.07 \]

Thus \[ \therefore EI \theta_B = 40.88, \ EI \theta_C = -40.88, \ EI \delta = 95.07 \]

Substituting these values in slope deflection equations

\[ M_{AB} = \frac{2}{3} (40.88) - \frac{2}{3} (95.07) = -36.12 \text{ KNM} \]
\[ M_{BA} = \frac{4}{3} (40.88) - \frac{2}{3} (95.07) = -8.88 \text{ KNM} \]
\[ M_{BC} = -32 + 2(40.88) + (-40.88) = 8.88 \text{ KNM} \]
\[ M_{CB} = +32 + 2(-40.88) + (40.88) = -8.88 \text{ KNM} \]
\[ M_{CD} = \frac{4}{3} (-40.88) + \frac{2}{3} (95.07) = +8.88 \text{ KNM} \]
\[ M_{DC} = \frac{2}{3} (-40.88) + \frac{2}{3} (95.07) = +36.12 \text{ KNM} \]
To find the reaction consider the free body diagram of the frame

Reactions:

Column AB

\[ H_A = \frac{8.88 + 36.12}{3} = 15 \text{ KN} \]

Beam AB

\[ R_B = \frac{+8.88 - 8.88 + 24 \times 4 \times \frac{4}{2}}{4} = 48 \text{ KN} \]

\[ \therefore R_C = 24 \times 4 - 48 = 48 \text{ KN} \]

Column CD
\[ H_D = \frac{8.88 + 36.12}{3} = 15 \text{ KN} \]

Check

\[ \Sigma H = 0 \]
\[ H_A + H_D + P = 0 \]
\[ -15 - 15 + 30 = 0 \]

Hence okay

V-UNIT MOMENT DISTRIBUTION METHOD

Advantages of Fixed Ends or Fixed Supports

1. Slope at the ends is zero.
2. Fixed beams are stiffer, stronger and more stable than SSB.
3. In case of fixed beams, fixed end moments will reduce the BM in each section.
4. The maximum deflection is reduced.

MOMENT DISTRIBUTION METHOD

INTRODUCTION AND BASIC PRINCIPLES

Introduction

(Method developed by Prof. Hardy Cross in 1932)

The method solves for the joint moments in continuous beams and rigid frames by successive approximation

Statement of Basic Principles

Consider the continuous beam ABCD, subjected to the given loads, as shown in Figure below. Assume that only rotation of joints occur...
at B, C and D, and that no support displacements occur at B, C and D. Due to the applied loads in spans AB, BC and CD, rotations occur at B, C and D.

In order to solve the problem in a successively approximating manner, it can be visualized to be made up of a continued two-stage problems viz., that of locking and releasing the joints in a continuous sequence. The joints B, C and D are locked in position before any load is applied on the beam ABCD; then given loads are applied on the beam. Since the joints of beam ABCD are locked in position, beams AB, BC and CD acts as individual and separate fixed beams, subjected to the applied loads; these loads develop fixed end moments.

In beam AB
Fixed end moment at A = \(-\frac{wl^2}{12}\) = \(-\frac{(15)(8)(8)}{12}\) = - 80 kN.m
Fixed end moment at B = \(\frac{wl^2}{12}\) = \(\frac{(15)(8)(8)}{12}\) = + 80 kN.m

In beam BC
Fixed end moment at B = \(-\frac{(P_{ab}a^2)}{l^2}\) = \(-\frac{(150)(3)(3)}{6}\)
= -112.5 kN.m
Fixed end moment at C = \(\frac{(P_{ab}a^2)}{l^2}\) = \(\frac{(150)(3)(3)}{6}\)
= + 112.5

In beam AB
Fixed end moment at C = \(-\frac{wl^2}{12}\) = \(-\frac{(10)(8)(8)}{12}\) = - 53.33 kN.m
Fixed end moment at D = \(\frac{wl^2}{12}\) = \(\frac{(10)(8)(8)}{12}\) = + 53.33 kN.m

Since the joints B, C and D were fixed artificially (to compute the fixed-end moments), now the joints B, C and D are released and allowed to rotate. Due to the joint release, the joints rotate maintaining the continuous nature of the beam. Due to the joint release, the fixed end moments on either side of joints B, C and D act in the opposite direction now, and cause a net unbalanced moment to occur at the joint.
These unbalanced moments act at the joints and modify the joint moments at B, C and D, according to their relative stiffnesses at the respective joints. The joint moments are distributed to either side of the joint B, C or D, according to their relative stiffnesses. These distributed moments also modify the moments at the opposite side of the beam span, viz., at joint A in span AB, at joints B and C in span BC and at joints C and D in span CD. This modification is dependent on the carry-over factor (which is equal to 0.5 in this case);
The carry-over moment becomes the unbalanced moment at the joints to which they are carried over. Steps 3 and 4 are repeated till the carry-over or distributed moment becomes small.

**Sum up all the moments at each of the joint** to obtain the joint moments.

**SOME BASIC DEFINITIONS**

In order to understand the five steps mentioned in section 7.3, some words need to be defined and relevant derivations made.

**7.3.1 Stiffness and Carry-over Factors**

Stiffness = Resistance offered by member to a unit displacement or rotation at a point, for given support constraint conditions

---

E, I – Member properties

A clockwise moment $M_A$ is applied at A to produce a +ve bending in beam AB. Find $\theta_A$ and $M_B$.

**Using method of consistent deformations**
Considering moment \( M_B \),
\[
M_B + M_A + R_AL = 0
\]
\[
\therefore M_B = M_A/2 = (1/2)M_A
\]

**Carry - over Factor = 1/2**

### 7.3.2 Distribution Factor

Distribution factor is the ratio according to which an externally applied unbalanced moment \( M \) at a joint is apportioned to the various members mating at the joint.

\[
\Delta_A = \frac{M_B L^2}{2EI}
\]

\[
f_{AA} = \frac{L^3}{3EI}
\]

Applying the principle of consistent deformation,
\[
\Delta_A + R_f f_{AA} = 0 \rightarrow R_f = \frac{3M_A}{2L}
\]

\[
\theta_A = \frac{M_A L}{EI} + \frac{R_AL^2}{2EI} = \frac{M_N L}{4EI}
\]

\[
\therefore M_A = \frac{4EI}{L} \theta_A \quad \text{hence} \quad k_{\theta} = \frac{M_A}{\theta_A} = \frac{4EI}{L}
\]

**Stiffness factor = \( k_{\theta} = 4EI/L \)**

**Considering moment \( M_B \),**
\[
M_B + M_A + R_AL = 0
\]
\[
\therefore M_B = M_A/2 = (1/2)M_A
\]

### Carry - over Factor = 1/2

**7.3.2 Distribution Factor**

Distribution factor is the ratio according to which an externally applied unbalanced moment \( M \) at a joint is apportioned to the various members mating at the joint.

At joint B
\[
M - M_{BA} - M_{BC} - M_{BD} = 0
\]

\[
M = M_{BA} + M_{BC} + M_{BD}
\]

\[
= \left[ \frac{4EI_1}{L_1} + \frac{4EI_2}{L_2} + \frac{4EI_3}{L_3} \right] \theta_B
\]

\[
= \left( K_{BA} + K_{BC} + K_{BD} \right) \theta_B
\]

\[
\therefore \theta_B = \frac{M}{K_{BA} + K_{BC} + K_{BD}} = \frac{M}{\sum K}
\]

\[
M_{BA} = K_{BA} \theta_B = \left( \frac{K_{BA}}{\sum K} \right) M = (D.F.)_{BA} M
\]
Modified Stiffness Factor

The stiffness factor changes when the far end of the beam is simply-supported.

As per earlier equations for deformation, given in Mechanics of Solids text-books.

\[ \theta_A = \frac{M_A L}{3EI} \]
\[ K_{AB} = \frac{M_A}{\theta_A} = \frac{3EI}{L} = \left( \frac{3}{4} \right) \left( \frac{4EI}{L} \right) = \frac{3}{4} (K_{AB})_{fixed} \]

Solve the previously given problem by the moment distribution method

**Fixed end moments**

\[ M_{AB} = -M_{BA} = -\frac{wl^2}{12} = -\frac{(15)(8)^2}{12} = -80 \text{ kN} \cdot \text{m} \]
\[ M_{BC} = -M_{CB} = -\frac{wl}{8} = -\frac{(150)(6)}{8} = -112.5 \text{ kN} \cdot \text{m} \]
\[ M_{CD} = -M_{DC} = -\frac{wl^2}{12} = -\frac{(10)(8)^2}{12} = -53.333 \text{ kN} \cdot \text{m} \]

**Stiffness Factors (Unmodified Stiffness)**

\[ K_{AB} = K_{BA} = \frac{4EI}{L} = \frac{(4)(EI)}{8} = 0.5EI \]
\[ K_{BC} = K_{CB} = \frac{4EI}{L} = \frac{(4)(EI)}{6} = 0.667EI \]
\[ K_{CD} = \left[ \frac{4EI}{8} \right] = \frac{4}{8} EI = 0.5EI \]
Distribution Factors

\[ DF_{AB} = \frac{K_{BA}}{K_{BA} + K_{wall}} = \frac{0.5EI}{0.5 + \infty} = 0.0 \]

\[ DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{0.5EI}{0.5EI + 0.667EI} = 0.4284 \]

\[ DF_{BC} = \frac{K_{BC}}{K_{BC} + K_{BC}} = \frac{0.667EI}{0.667EI + 0.500EI} = 0.5716 \]

\[ DF_{CB} = \frac{K_{CB}}{K_{CB} + K_{CD}} = \frac{0.667EI}{0.667EI + 0.500EI} = 0.5716 \]

\[ DF_{CD} = \frac{K_{CD}}{K_{CD}} = \frac{0.500EI}{0.667EI + 0.500EI} = 0.4284 \]

\[ DF_{DC} = \frac{K_{DC}}{K_{DC}} = 1.00 \]

Moment Distribution Table

<table>
<thead>
<tr>
<th>Joint</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Member</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
<td>CB</td>
</tr>
<tr>
<td>Distribution Factors</td>
<td>0</td>
<td>0.4284</td>
<td>0.5716</td>
<td>0.64</td>
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<tr>
<td>Cycle 1 Computed end moments</td>
<td>-80</td>
<td>80</td>
<td>-112.5</td>
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<tr>
<td>Distribution</td>
<td>13.923</td>
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<td>-37.87</td>
<td>-21.3</td>
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<td>Cycle 2 Distribution</td>
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<td>11.122</td>
<td>6.256</td>
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<tr>
<td>Carry-over moments</td>
<td>4.056</td>
<td>5.561</td>
<td>5.412</td>
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<td>Carry-over moments</td>
<td>-1.191</td>
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<td>Cycle 4 Distribution</td>
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<td>1.964</td>
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<td>Carry-over moments</td>
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<td>1.01</td>
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<td>Cycle 5 Distribution</td>
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<td>-0.702</td>
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<tr>
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<td>75</td>
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<tr>
<td>End reaction</td>
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<tr>
<td>due to left hand FEM</td>
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<td>8.726</td>
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<tr>
<td>End reaction</td>
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<tr>
<td>due to right hand FEM</td>
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<tr>
<td>Summed-up moments</td>
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<td>56.228</td>
<td>63.772</td>
<td>75.563</td>
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</tbody>
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